

# Duality, Monopoles, Dyons, Confinement and Oblique Confinement in Supersymmetric $SO(N_c)$ Gauge Theories

K. Intriligator<sup>1</sup> and N. Seiberg<sup>1,2</sup>

<sup>1</sup>Department of Physics and Astronomy

Rutgers University

Piscataway, NJ 08855-0849, USA

<sup>2</sup> School of Natural Sciences

Institute for Advanced Study

Princeton, NJ 08540, USA

We study supersymmetric  $SO(N_c)$  gauge theories with  $N_f$  flavors of quarks in the vector representation. Among the phenomena we find are dynamically generated superpotentials with physically inequivalent branches, smooth moduli spaces of vacua, confinement and oblique confinement, confinement without chiral symmetry breaking, massless composites (glueballs, exotics, monopoles and dyons), non-trivial fixed points of the renormalization group and massless magnetic quarks and gluons. Our analysis sheds new light on a recently found duality in  $N = 1$  supersymmetric theories. The dual forms of some of the theories exhibit “quantum symmetries” which involve non-local transformations on the fields. We find that in some cases the duality has both  $S$  and  $T$  transformations generating  $SL(2, Z)$  (only an  $S_3$  quotient of which is realized non-trivially). They map the original non-Abelian electric theory to magnetic and dyonic non-Abelian theories. The magnetic theory gives a weak coupling description of confinement while the dyonic theory gives a weak coupling description of oblique confinement. Our analysis also shows that the duality in  $N = 1$  is a generalization of the Montonen-Olive duality of  $N = 4$  theories.

## 1. Introduction

Recently, it has become clear that certain aspects of four dimensional supersymmetric field theories can be analyzed exactly, thus providing a laboratory for the analysis of the dynamics of gauge theories [1-15] (for a recent short review, see [16]). Most of the work so far was devoted to  $SU(N_c)$  gauge theories (see [17-20] for earlier work on these theories), which exhibited interesting physical phenomena. It is natural to ask which of these results are specific to  $SU(N_c)$  and which are more general. Furthermore, other theories might exhibit qualitatively new phenomena. Here we address these questions by studying  $SO(N_c)$  gauge theories with  $N_f$  flavors of quarks,  $Q^i$  ( $i = 1, \dots, N_f$ ), in the vector representation of the gauge group. Another motivation to study these theories is associated with the role of the center of the group in confinement. Unlike the  $SU(N_c)$  theories with quarks in the fundamental representation, where there is no invariant distinction between Higgs and confinement [21], here these phases (as well as the oblique confinement phase [22,23]) can be distinguished. A Wilson loop in the spinor representation cannot be screened by the dynamical quarks and therefore it is a gauge invariant order parameter for confinement. A similar comment applies to the 'tHooft loop and to the product of the 'tHooft loop and the Wilson loop (which probes oblique confinement).

Our results may be summarized as follows:

Theories with  $N_f < N_c - 4$  have dynamically generated superpotentials, associated with gaugino condensation, similar to the ones found in  $SU(N_c)$  theories with  $N_f < N_c$  [17].

Theories with  $N_f = N_c - 4$  have two physically inequivalent phase branches. One branch has a dynamically generated superpotential, just as with  $N_f < N_c - 4$ . On the other branch no superpotential is generated; on this branch there is a moduli space of physically inequivalent but degenerate quantum vacua. This moduli space of vacua differs from the classical one in that a classical singularity at the origin is smoothed out in the quantum theory in a fashion similar to  $SU(N_c)$  gauge theories with  $N_f = N_c$  [2] and to an  $SU(2)$  gauge theory with quarks in the  $I = 3/2$  representation [9]. The theory at the origin of this space has confinement without chiral symmetry breaking.

For  $N_f = N_c - 3$ , we again find two physically inequivalent phase branches, one with a dynamically generated superpotential and the other with a moduli space of quantum vacua. In the branch with the moduli space of vacua there is again confinement without chiral symmetry breaking and there are  $N_f$  massless composite fields at the origin. They

can be interpreted as glueballs (for  $N_c = 4$ ) or exotics (for  $N_c > 4$ ). This phenomenon of massless composites is similar to the one found in  $SU(N_c)$  theories with  $N_f = N_c + 1$  [2].

Theories with  $N_f = N_c - 2$  have no superpotential – there is, again, a quantum moduli space of degenerate vacua. The low energy theory contains a massless photon and hence the theory is in a Coulomb phase. As in [5-8] we exactly compute its effective gauge coupling on the quantum moduli space of vacua. We find various numbers of massless monopoles and dyons at various vacua at strong  $SO(N_c)$  coupling. As in [6], they transform non-trivially under the flavor symmetry. When a mass term is added the monopoles and dyons condense leading to confinement and oblique confinement [22,23] respectively. These correspond to the two physically inequivalent branches of the theories with  $N_f = N_c - 3, N_c - 4$ .

For  $N_f > N_c - 2$ , the theory at the origin of the space of vacua is in a non-Abelian Coulomb phase. It can be given a dual “magnetic” description in terms of an  $SO(N_f - N_c + 4)$  theory with matter discussed in [10]. The dynamical scale  $\tilde{\Lambda}$  of the dual theory is inversely related to the scale  $\Lambda$  of the original theory

$$\Lambda^{3(N_c-2)-N_f} \tilde{\Lambda}^{2N_f-3(N_c-2)} \sim \mu^{N_f} \quad (1.1)$$

and therefore the electric theory becomes weaker as the magnetic theory becomes stronger and vice versa. The meaning of this relation and of the scale  $\mu$  will be explained in detail. The interpretation of the dual theory as being “magnetic” becomes obvious in the special case  $N_f = N_c - 2$  where the low energy gauge group is  $U(1)$  and the dual matter fields are the magnetic monopoles. The duality for  $N_f > N_c - 2$  is then clearly identified as a non-Abelian generalization of ordinary electric-magnetic duality. For  $N_c - 2 < N_f \leq \frac{3}{2}(N_c - 2)$  the magnetic degrees of freedom are free in the infra-red while for  $\frac{3}{2}(N_c - 2) < N_f < 3(N_c - 2)$  the electric and the magnetic theories flow to the same non-trivial fixed point of the renormalization group.

In sections 2 – 4 we discuss all these cases. We start from a small number of flavors and gradually consider larger  $N_f$ . We then check that our results fit together upon giving the quarks  $Q^i$  masses and reducing the number of flavors.

One lesson from these theories is that the qualitative phenomena found in  $SU(N_c)$  theories and some  $N = 2$  theories are more generic and apply in a wider class of  $N = 1$  theories. Other lessons are associated with the new subtleties which are specific to these theories.

In section 5 we discuss  $SO(3)$  gauge theories with  $N_f$  quarks. They exhibit new complications which are not present for larger values of  $N_c$ . Some of their dual theories exhibit quantum symmetries. These are symmetries which are not easily visible from the Lagrangian because they are implemented by non-local transformations on the fields.

The  $SO(3)$  theories lead us to the first example of new duality transformations in  $N = 1$  theories. The electric theory can be transformed both to a magnetic and to a dyonic theory. The electric theory is weakly coupled in the Higgs branch of the theory (along the flat directions) and strongly coupled in the confining and the oblique confining branches of the theory. The magnetic (dyonic) theory is weakly coupled in the confining (oblique confinement) branch of the theory and is strongly coupled in the Higgs and the oblique confinement (confinement) branches. The confining and the oblique confinement branches of the theory are related by a spontaneously broken global discrete symmetry. Therefore, the magnetic and the dyonic theories are similar. The group of duality transformations which permutes these theories is  $S_3$ . It is related to the standard duality group  $SL(2, Z)$  by a quotient by  $\Gamma(2)$ , which acts trivially on the theories. In other words, the duality transformation  $S \in SL(2, Z)$  relates the electric theory to the magnetic theory while  $T \in SL(2, Z)$  maps the magnetic theory to the dyonic theory.

The analysis of  $SO(3)$  with  $N_f = 3$  establishes the relation between the Montonen-Olive duality [24] of  $N = 4$  theories [25] and the duality in  $N = 1$  theories [10]. When a generic cubic superpotential is turned on the theory flows in the infra-red to an  $N = 4$  theory. Its  $N = 1$  dual is an  $SO(4)$  theory, with  $N_f = 3$ , which flows in the infra-red to an  $SU(2)$   $N = 4$  theory<sup>1</sup>. These two similar  $N = 4$  theories are dual to each other as in [24,25]. Therefore, the duality in  $N = 1$  theories [10] is compatible with and generalizes the Montonen-Olive duality of  $N = 4$ .

In section 6 we present more dyonic theories for  $N_f = N_c - 1$ . Unlike the dyonic theories discussed in section 5, here there is no global symmetry which makes the theories similar. The electric, magnetic and dyonic theories are really distinct. As in section 5, the electric theory gives a weak coupling description of the Higgs branch of the theory, the magnetic theory gives a weak coupling description of the confining branch and the dyonic theory gives a weak coupling description of the oblique confinement branch of the theory. In all these examples the magnetic theory is weakly coupled at the non-Abelian Coulomb point while the other two theories are strongly coupled there.

---

<sup>1</sup> Throughout this paper we limit ourselves to the Lie algebra and do not discuss the role of the center of the gauge group. Therefore we do not distinguish between  $SO(N_c)$  and  $Spin(N_c)$ .

## 2. Preliminaries

### 2.1. The classical moduli spaces for $SO(N_c)$ with $N_f$ quarks $Q^i$ .

The classical theory with  $N_f$  massless quarks has a moduli space of degenerate vacua labeled by the expectation values  $\langle Q \rangle$  of the scalar components of the matter fields subject to the “D-flatness” constraints. Up to gauge and global rotations, the solutions of these equations for  $N_f < N_c$  are of the form

$$Q = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{N_f} \end{pmatrix} \quad (2.1)$$

where, using gauge transformations, the sign of any  $a_i$  can be flipped. For generic  $a_i$  these expectation values break  $SO(N_c)$  to  $SO(N_c - N_f)$  by the Higgs mechanism for  $N_f \leq N_c - 2$  and completely break  $SO(N_c)$  for  $N_f > N_c - 2$ . For  $N_f \geq N_c$  the flat directions are of the form

$$Q = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{N_c} \end{pmatrix}. \quad (2.2)$$

If some of the  $a_i$  vanish, the signs of the others can be flipped by gauge transformations. However, if all  $a_i \neq 0$ , gauge transformations can only be used to flip signs in such a way that the product  $\prod_{i=1}^{N_c} a_i$  is invariant.

For  $N_f < N_c$  the space of vacua can be given a gauge invariant description in terms of the expectation values of the “meson” fields  $M^{ij} = Q^i \cdot Q^j$ . These  $\frac{1}{2}N_f(N_f + 1)$  superfields correspond precisely to the matter superfields left massless after the Higgs mechanism. Their expectation values are classically unconstrained. For  $N_f \geq N_c$  it is also possible to form “baryons”  $B^{[i_1 \dots i_{N_c}]} = (Q)^{N_c}$ , with indices antisymmetrized. Along the flat direction (2.2) we have

$$M = \begin{pmatrix} a_1^2 & & & \\ & a_2^2 & & \\ & & \ddots & \\ & & & a_{N_c}^2 \end{pmatrix} \quad (2.3)$$

$$B^{1, \dots, N_c} = a_1 \dots a_{N_c}$$

with all other components of  $M$  and  $B$  vanishing.  $M$  thus has rank at most  $N_c$ . If the rank of  $M$  is less than  $N_c$ ,  $B = 0$ . If the rank of  $M$  is  $N_c$ ,  $B$  has rank one and its non-zero eigenvalue is, with an undetermined sign, the square root of the product of non-zero eigenvalues of  $M$ . The classical moduli space of vacua for  $N_f \geq N_c$  is therefore described by the space of  $M$  of rank at most  $N_c$  along with a sign, corresponding to the sign of  $B = \pm\sqrt{\det' M}$ , for  $M$  of rank  $N_c$ .

## 2.2. The quantum theories and some conventions

The quantum  $SO(N_c)$  theory with  $N_f$  massless quarks has an anomaly free global  $SU(N_f) \times U(1)_R$  symmetry with the fields  $Q$  transforming as  $(N_f)_{\frac{N_f - N_c + 2}{N_f}}$ . In addition, the theory is invariant under the  $Z_2$  charge conjugation symmetry  $\mathcal{C}$  and the  $Z_{2N_f}$  ( $Z_{4N_f}$  for  $N_c = 3$ ) discrete symmetry

$$\begin{aligned} Q &\rightarrow e^{2\pi i/2N_f} Q \\ Q &\rightarrow e^{2\pi i/4N_f} Q \quad \text{for } N_c = 3 \end{aligned} \tag{2.4}$$

(its  $Z_{N_f}$  subgroup is also in the center of  $SU(N_f)$ ).

For  $N_c > 4$  the one-loop beta function implies that the gauge coupling runs as  $e^{-8\pi^2 g^{-2}(E) + i\theta} = (\Lambda_{N_c, N_f}/E)^{3(N_c-2)-N_f}$ , where  $\Lambda_{N_c, N_f}$  is the dynamically generated scale for the theory with  $N_f$  quarks. By adding the  $\theta$  angle, the scale  $\Lambda$  becomes a complex number, which can be interpreted as the first component of a chiral superfield. Since  $SO(4) \cong SU(2)_L \times SU(2)_R$ , there are independent running gauge couplings for each  $SU(2)_s$ :  $e^{-8\pi^2 g_s^{-2}(E) + i\theta} = (\Lambda_{s, N_f}/E)^{6-N_f}$  for  $s = L, R$ . For  $N_c = 3$  the gauge coupling runs as  $e^{-8\pi^2 g^{-2}(E) + i\theta} = (\Lambda_{3, N_f}/E)^{6-2N_f}$ .

By giving the quarks  $Q^{N_f}$  a mass term with  $W_{tree} = \frac{1}{2}mM^{N_f N_f}$  and decoupling the massive matter, the theory with  $N_f$  quarks yields a low energy theory with  $N_f - 1$  quarks. Matching the running gauge coupling at the mass scales where the massive quarks decouple relates the scale of the original high-energy theory to the scale of the low energy Yang-Mills theory as

$$\begin{aligned} \Lambda_{N_c, N_f}^{3(N_c-2)-N_f} m &= \Lambda_{N_c, N_f-1}^{3(N_c-2)-(N_f-1)} & \text{for } N_c > 4 \\ \Lambda_{s, N_f}^{6-N_f} m &= \Lambda_{s, N_f-1}^{6-(N_f-1)} & \text{for } N_c = 4, \quad s = L, R. \\ \Lambda_{3, N_f}^{6-2N_f} m^2 &= \Lambda_{3, N_f-1}^{6-2(N_f-1)} & \text{for } N_c = 3 \end{aligned} \tag{2.5}$$

The absence of any constant “threshold” factors in these matching relations define our normalization for  $\Lambda_{N_c, N_f}$  relative to the normalization of  $\Lambda_{N_c, 0}$ . (For  $N_c = 3$  this convention differs slightly from the one used in [7].) For more discussion on the threshold factors in these theories see [15].

The  $SO(N_c)$  theory with  $N_f$  quarks can also be related to an  $SO(N_c - 1)$  theory with  $N_f - 1$  quarks via the Higgs mechanism by taking the expectation value  $a_{N_f}$  in (2.1) to be large. The scale  $\Lambda_{N_c-1, N_f-1}$  of the low energy theory is related to the scale  $\Lambda_{N_c, N_f}$  of the original theory by matching the running gauge coupling at the energy scale set by  $a_{N_f}$ ,

$$\begin{aligned} \Lambda_{N_c, N_f}^{3(N_c-2)-N_f} (M^{N_f N_f})^{-1} &= \Lambda_{N_c-1, N_f-1}^{3(N_c-2)-N_f-2} && \text{for } N_c > 5 \\ \Lambda_{5, N_f}^{9-N_f} (M^{N_f N_f})^{-1} &= \Lambda_{s, N_f-1}^{6-(N_f-1)} && \text{for } s = L, R \\ 4\Lambda_{L, N_f}^{6-N_f} \Lambda_{R, N_f}^{6-N_f} (M^{N_f N_f})^{-2} &= \Lambda_{3, N_f-1}^{6-2(N_f-1)}. \end{aligned} \quad (2.6)$$

For  $N_c = 4$ , the quarks are in the  $(2, 2)$  representation of  $SU(2)_L \times SU(2)_R$  and we have  $M^{N_f N_f} = Q^{N_f} \cdot Q^{N_f} = Q_{c_L c_R}^{N_f} Q_{d_L d_R}^{N_f} \epsilon^{c_L d_L} \epsilon^{c_R d_R}$ . The factor of four in the last relation of (2.6) reflects the fact that the natural order parameter in terms of the  $SU(2)_s$  is  $\frac{1}{2} M^{N_f N_f}$ . These relations define our relative normalization of the  $\Lambda_{N_c, N_f}$  for different  $N_c$ . To fix the absolute normalization, we use the conventions of [4] (see also [15]).

With the scale normalizations defined above, gaugino condensation in the pure  $SO(N_c)$  Yang-mills theory is found to be given by

$$\begin{aligned} \langle \lambda \lambda \rangle &= \frac{1}{2} 2^{4/(N_c-2)} \epsilon_{N_c-2} \Lambda_{N_c, 0}^3 && \text{for } N_c \geq 5 \\ \langle (\lambda \lambda)_s \rangle &= \epsilon_s \Lambda_{s, 0}^3 && \text{for } N_c = 4, \ s = L, R \\ \langle \lambda \lambda \rangle &= \epsilon_2 \Lambda_{3, 0}^3 && \text{for } N_c = 3, \end{aligned} \quad (2.7)$$

where  $\epsilon_{N_c-2}$  is an  $(N_c - 2)$ -th root of unity, reflecting the  $N_c - 2$  (physically equivalent) supersymmetric vacua of the pure gauge supersymmetric  $SO(N_c)$  theory and, likewise,  $\epsilon_s = \pm 1$  for  $s = L, R$  and  $\epsilon_2 = \pm 1$ . The last two equations follow the convention for  $SU(2)$  of [4]. The first one can be derived by studying the theory with matter and perturbing it with mass terms or along the flat directions as in section 3 (see also [15]).

Throughout most of this paper, we will limit the discussion of  $SO(4)$  to the case  $\Lambda_L = \Lambda_R$ , which is similar to larger values of  $N_c$ .

The classical vacuum degeneracy of (2.1) and (2.2) can be lifted by quantum effects. In the low energy effective theory this is represented by a dynamically generated superpotential for the light meson fields  $M^{ij}$ . Holomorphy and the  $SU(N_f) \times U(1)_R$  symmetries determine that any dynamically generated superpotential (for  $N_c \neq 3$ ) must be of the form

$$W = A_{N_c, N_f} \left( \frac{\Lambda_{N_c, N_f}^{3N_c - N_f - 6}}{\det_{ij}(Q^i \cdot Q^j)} \right)^{1/(N_c - N_f - 2)} = A_{N_c, N_f} \left( \frac{\Lambda_{N_c, N_f}^{3N_c - N_f - 6}}{\det M} \right)^{1/(N_c - N_f - 2)}, \quad (2.8)$$

for some constants  $A_{N_c, N_f}$ . For  $N_f = N_c - 2$  the superpotential (2.8) does not make sense. For  $N_c - 2 < N_f \leq N_c$ , the superpotential (2.8) cannot be generated because it has incorrect asymptotic behavior in the limit of large  $|Q|$ , where asymptotic freedom implies that any dynamically generated superpotential must be smaller than  $|Q|^3$ . For  $N_f > N_c$ ,  $\det Q^i \cdot Q^j = 0$  and so the superpotential (2.8) cannot exist. In short, there can be no dynamically generated superpotential for  $N_f \geq N_c - 2$ . Similarly, (in the Higgs phase) there can be no dynamically generated superpotential for  $N_c = 3$  for any  $N_f$ . These theories have a quantum moduli space of exactly degenerate but physically inequivalent vacua. We find that these theories display other interesting non-perturbative gauge dynamics. Even for  $N_f < N_c - 2$  where a consistent invariant superpotential exists, we will show that it is not always generated.

### 3. The quantum theories for $N_c \geq 4$ , $N_f \leq N_c - 2$

#### 3.1. $N_f \leq N_c - 5$ ; a dynamically generated superpotential by gaugino condensation.

As in [17], the superpotential (2.8) is generated by gaugino condensation in the  $SO(N_c - N_f)$  Yang-Mills theory left unbroken by the  $\langle Q \rangle$ :  $W = (N_c - N_f - 2)\langle \lambda\lambda \rangle$ ; the details leading to the normalization factor were discussed in [4] for  $SU(N_c)$  gaugino condensation. Following the conventions discussed in the previous section, this gives

$$W = \frac{1}{2}(N_c - N_f - 2)\epsilon_{(N_c - N_f - 2)} \left( \frac{16\Lambda_{N_c, N_f}^{3N_c - N_f - 6}}{\det M} \right)^{1/(N_c - N_f - 2)}. \quad (3.1)$$

This quantum effective superpotential lifts the classical vacuum degeneracy. Indeed, the theory (3.1) has no vacuum at all. Adding mass terms  $W_{tree} = \frac{1}{2}\text{Tr } mM$  to the dynamically generated superpotential (3.1) gives a theory with  $(N_c - 2)$  supersymmetric vacua:

$$\langle M^{ij} \rangle = \epsilon_{(N_c - 2)} \left( 16 \det m \Lambda_{N_c, N_f}^{3(N_c - 2) - N_f} \right)^{1/(N_c - 2)} \left( \frac{1}{m} \right)^{ij}. \quad (3.2)$$

If some of the masses are zero, we can integrate out the massive quarks to find an effective superpotential for the massless ones. It is of the form (3.1), with the scale of the low energy theory with fewer quarks given by (2.5).



3.2.  $N_f = N_c - 4$ ; two inequivalent branches – confinement without chiral symmetry breaking

In this case the  $\langle Q \rangle$  break  $SO(N_c)$  to  $SO(4) \cong SU(2)_L \times SU(2)_R$ . With the conventions discussed above, the scales  $\Lambda_{s,0}$  of the low energy  $SU(2)_s$  Yang-Mills theories are related to the scale of the high energy theory by  $\Lambda_{L,0}^6 = \Lambda_{R,0}^6 = \Lambda_{N_c, N_c-4}^{2(N_c-1)} / \det M$ . Gaugino condensation in the unbroken  $SU(2)_L \times SU(2)_R$  generates the superpotential

$$W = 2\langle\lambda\lambda\rangle_L + 2\langle\lambda\lambda\rangle_R = \frac{1}{2}(\epsilon_L + \epsilon_R) \left( \frac{16\Lambda_{N_c, N_c-4}^{2(N_c-1)}}{\det M} \right)^{1/2}, \quad (3.3)$$

where  $\epsilon_L$  and  $\epsilon_R$  are  $\pm 1$  and the factors of two follow from the discussion in [4].

The  $\epsilon_s$  in (3.3) reflect the fact that the low energy theory has four ground states. The two ground states with  $\epsilon_L = \epsilon_R$  are physically equivalent; they are related by a discrete R symmetry. The two ground states with  $\epsilon_L = -\epsilon_R$  are also physically equivalent. However, the ground states with  $\epsilon_L = \epsilon_R$  are physically distinct from those with  $\epsilon_L = -\epsilon_R$ . The sign of  $\epsilon_L \epsilon_R$  labels two physically inequivalent phase branches of the low energy effective theory. The branch of (3.3) with  $\epsilon_L \epsilon_R = 1$  is simply the continuation of (3.1) to  $N_f = N_c - 4$ . It lifts the classical vacuum degeneracy and the quantum theory has no vacuum.

The two ground states with  $\epsilon_L \epsilon_R = -1$  are different. The superpotential (3.3) is then zero; there is a quantum moduli space of degenerate but physically inequivalent vacua labeled by  $\langle M \rangle$  (the two different values of  $\epsilon_L$  on this branch mean that for every  $\langle M \rangle$  there are two ground states). In particular, there is a vacuum at the origin,  $M = 0$ . Classically, the low energy effective theory has a singularity at the origin, corresponding to the  $SO(N_c)/SO(4)$  vector bosons which become massless there. This singularity shows up in the classical Kahler potential  $K_{classical}(M, M^\dagger)$ . In the quantum theory such a singularity is either smoothed out or it is associated with some fields which become massless there. Our result is that the classical singularity at the origin is simply smoothed out. In other words, the massless spectrum at the origin is the same as it is elsewhere, consisting simply of the fields  $M$ .

This result satisfies several independent and highly non-trivial consistency conditions. For example, because the theory has a global  $SU(N_f) \times U(1)_R$  symmetry which is unbroken at the origin, this assertion about the massless spectrum at the origin can be checked using the 't Hooft anomaly matching conditions. The classical massless fermions are the quark components of the  $Q^i$ , with the global quantum numbers  $N_c \times (N_f)_{\frac{2-N_c}{N_f}}$ , and the gluinos

with the global quantum numbers  $\frac{1}{2}N_c(N_c - 1) \times (1)_1$ . This classical massless spectrum gives for the 't Hooft anomalies

$$\begin{aligned}
U(1)_R & -\frac{1}{2}N_c(N_c - 3) \\
U(1)_R^3 & \frac{1}{2}N_c(N_c - 1) + \frac{N_c}{N_f^2}(2 - N_c)^3 \\
SU(N_f)^3 & N_c d_3(N_f) \\
SU(N_f)^2 U(1)_R & N_c \left(\frac{2 - N_c}{N_f}\right) d_2(N_f),
\end{aligned} \tag{3.4}$$

where  $d_2(N_f)$  and  $d_3(N_f)$  are the quadratic and cubic  $SU(N_f)$  Casimirs in the fundamental representation. Our asserted massless fermionic spectrum at the origin in the quantum theory is simply the fermionic component of  $M$ , with the global quantum numbers  $(\frac{1}{2}N_f(N_f + 1))_{\frac{N_f - 2N_c + 4}{N_f}}$ ; this field has the 't Hooft anomalies

$$\begin{aligned}
U(1)_R & \frac{1}{2}(N_f + 1)(N_f - 2N_c + 4) \\
U(1)_R^3 & \frac{1}{2}N_f^{-2}(N_f + 1)(N_f - 2N_c + 4)^3 \\
SU(N_f)^3 & (N_f + 4)d_3(N_f) \\
SU(N_f)^2 U(1)_R & (N_f + 2)\left(\frac{N_f - 2N_c + 4}{N_f}\right)d_2(N_f).
\end{aligned} \tag{3.5}$$

It is non-trivial but true that these anomalies match the anomalies (3.4) of the microscopic theory for  $N_f = N_c - 4$ . As in [9], we use this fact as evidence that the Kahler potential near the origin is smoothed out by quantum effects:  $K(M \rightarrow 0) \sim \text{Tr } M^\dagger M / |\Lambda|^2$ .

Consider now giving  $Q^{N_f}$  a mass and integrating it out. The resulting low energy theory should agree with our prior results for  $N_f = N_c - 5$ . For the branch of (3.3) with  $\epsilon_L \epsilon_R = 1$ , adding  $W_{tree} = \frac{1}{2}mM^{N_f N_f}$  and integrating out the massive fields indeed gives (3.1). The branch with  $\epsilon_L \epsilon_R = -1$  has  $W = \frac{1}{2}mM^{N_f N_f}$ . As in [9], because the Kahler potential is everywhere smooth in  $M$ , this branch does not give a supersymmetric ground state. Therefore, the branch with  $\epsilon_L \epsilon_R = -1$  is properly eliminated from the effective low energy theories with  $N_f \leq N_c - 5$ .

Had there been new massless states somewhere on the branch with  $\epsilon_L \epsilon_R = -1$ , the addition of the  $Q^{N_f}$  mass term,  $W_{tree}$ , would have led to additional ground states. Since there are no such extra ground states for  $N_f < N_c - 4$ , we indeed conclude that the manifold of quantum vacua must be smooth and without any new massless fields.

The physics at  $\langle M \rangle = 0$  is interesting. Classically, there were massless quarks and gluons there. Quantum mechanically, only the  $M$  quanta are massless. This clearly signals the confinement of the elementary degrees of freedom. However, as is clear from the discussion above, the global chiral symmetry  $SU(N_f) \times U(1)_R$  is clearly unbroken. This is another example of the phenomenon observed in [2,6,9] of confinement without chiral symmetry breaking.

### 3.3. $N_f = N_c - 3$ ; two branches and massless composites

The expectation values  $\langle Q \rangle$  generically break  $SO(N_c)$  to  $SO(3)$ . The superpotential (2.8) can be found by examining the limit where the first  $N_f - 1$  eigenvalues of  $\langle M \rangle$  are large, breaking the theory to  $SU(2)_L \times SU(2)_R$  with one quark  $Q^{N_f}$ . Matching the running gauge coupling at the scales of the Higgs mechanism, the scales of the low energy  $SU(2)_L \times SU(2)_R$  theory are  $\Lambda_{L,1}^5 = \Lambda_{R,1}^5 = \Lambda_{N_c, N_c-3}^{2N_c-3} / \det \widehat{M}$ , where  $\det \widehat{M} = \det M / M^{N_f N_f}$ . The expectation value  $\langle Q^{N_f} \rangle$  breaks the  $SU(2)_L \times SU(2)_R$  gauge group of this low-energy theory to a diagonally embedded  $SO(3)$  with a scale  $\Lambda_D^6 = 4\Lambda_{L,1}^5 \Lambda_{R,1}^5 (M^{N_f N_f})^{-2}$ . Gaugino condensation in the unbroken  $SO(3)$  generates a superpotential  $W_D = 2\epsilon \Lambda_D^3$ , where  $\epsilon = \pm 1$ . In addition, as in [17], an instanton in the broken<sup>2</sup>  $SU(2)_L$  generates a superpotential  $W_L = 2\Lambda_{L,1}^5 / M^{N_f N_f}$  and an instanton in the broken  $SU(2)_R$  generates a superpotential  $W_R = 2\Lambda_{R,1}^5 / M^{N_f N_f}$ . Adding these three contributions and using the above matching relations, the superpotential (2.8) for  $SO(N_c)$  with  $N_f = N_c - 3$  quarks is

$$W = 4(1 + \epsilon) \frac{\Lambda_{N_c, N_c-3}^{2N_c-3}}{\det M}. \quad (3.6)$$

The low energy theory again has two physically inequivalent phase branches labeled by the sign of  $\epsilon$ . The branch with  $\epsilon = 1$  is the continuation of (3.1) to  $N_f = N_c - 3$ . The branch

---

<sup>2</sup> Typically when the gauge group  $G$  is broken to a non-Abelian subgroup  $H$  along the flat direction we do not need to consider instantons in the broken part of the group. The reason for that is that the phrase “instantons in the broken part of the group” is not well defined; these instantons can be rotated into  $H$ . Then, the strong dynamics in the low energy  $H$  gauge theory is stronger than these instanton effects. However, when the instantons in the broken part of the group are well defined, their effect must be taken into account when integrating out the massive gauge fields. This is the case when  $G$  (or one of its factors) is completely broken or broken to an Abelian subgroup, or when the index of the embedding of  $H$  in  $G$  is larger than one. In our case the index of the embedding is 2 and therefore we should include these instantons.

with  $\epsilon = -1$  has vanishing superpotential<sup>3</sup> and, therefore, has a quantum moduli space of vacua.

Upon adding a mass term  $W_{tree} = \frac{1}{2}mM^{N_f N_f}$  and integrating out  $Q^{N_f}$ , the branch of (3.6) with  $\epsilon = 1$  properly gives the two ground states of the  $\epsilon_L \epsilon_R = 1$  branch of (3.3). Upon adding  $W_{tree} = \frac{1}{2}mM^{N_f N_f}$  to the  $\epsilon = -1$  branch of the theory, we must likewise get the two ground states of the  $\epsilon_L \epsilon_R = -1$  branch of (3.3). In order for the  $\epsilon = -1$  branch of the theory to not be eliminated upon adding  $W_{tree}$ , there must be additional massless fields at the origin. Since they should not be present at generic points on the moduli space, there must be a superpotential giving them a mass away from  $M = 0$ . The simplest way to achieve that is to have fields,  $q_i$ , coupled to  $M$  with a superpotential which behaves as

$$W \approx \frac{1}{2\mu} M^{ij} q_i q_j \quad (3.7)$$

for  $M \approx 0$ , where  $\mu$  is a dimensionful scale needed if  $q$  has dimension one because  $M$  has dimension two. Adding  $W_{tree}$  to (3.7) and integrating out the massive fields indeed gives two physically equivalent ground states with  $W = 0$ , associated with the two sign choices in  $\langle q_{N_f} \rangle = \pm i\sqrt{m\mu}$ . These two ground states correspond to the two choices of  $\epsilon_1$  and  $\epsilon_2$  in the  $\epsilon_1 \epsilon_2 = -1$  branch of the low energy  $N_f = N_c - 4$  theory.

In order for (3.7) to respect the global flavor symmetry, the field  $q_i$  should have the  $SU(N_f) \times U(1)_R$  quantum numbers  $(\overline{N}_f)_{1+\frac{1}{N_f}}$ . The most general invariant superpotential is then

$$W = \frac{1}{2\mu} f \left( t = \frac{(\det M)(M^{ij} q_i q_j)}{\Lambda_{N_c, N_c-3}^{2N_c-2}} \right) M^{ij} q_i q_j. \quad (3.8)$$

In order for the superpotential (3.8) to yield the ground states discussed above, the function  $f(t)$  must be holomorphic in a neighborhood of  $t = 0$ . The  $q_i$  can be rescaled to set  $f(0) = 1$ .

The field  $q_i$  was motivated by requiring the correct behavior upon giving a flavor a mass and integrating it out. It is a highly non-trivial independent check that the 't Hooft anomalies with the massless spectrum at the origin consisting of the  $M^{ij}$  and  $q_i$  match the

---

<sup>3</sup> This happens as a result of cancellation between a high energy contribution (the term proportional to 1 in (3.6)) and a low energy contribution (the term proportional to  $\epsilon = -1$  in (3.6)). Can such a cancellation between high energy and low energy contributions, which does not follow from any symmetry, be relevant to the problem of the cosmological constant?

anomalies (3.4) of the classical spectrum. The fermion component of the field  $q_i$  gives the 't Hooft anomalies

$$\begin{aligned}
U(1)_R & 1 \\
U(1)_R^3 & N_f^{-2} \\
SU(N_f)^3 & -d_3(N_f) \\
SU(N_f)^2 U(1)_R & N_f^{-1} d_2(N_f).
\end{aligned} \tag{3.9}$$

Adding these to the contribution (3.5) of the field  $M$ , the anomalies associated with the massless spectrum  $M$  and  $q$  do indeed match the microscopic anomalies (3.4) for  $N_f = N_c - 3$ . The massless field  $q_i$  is naturally identified as  $q_i = \Lambda_{N_c, N_c-3}^{2-N_c} b_i$  where  $b_i$  is the “exotic” composite  $b_i = (Q)^{N_c-4} W_\alpha W^\alpha$ , (for  $N_c = 4$ , which we will discuss below, this is a glueball) with the color indices contracted with an epsilon tensor, as they have the same quantum numbers. In terms of  $b$ , whose dimension is  $N_c - 1$ , (3.8) is

$$W = \frac{1}{2\Lambda^{2N_c-3}} f \left( t = \frac{(\det M)(M^{ij} b_i b_j)}{\Lambda_{N_c, N_c-3}^{4N_c-6}} \right) M^{ij} b_i b_j \tag{3.10}$$

where we absorbed  $\mu$  in the definition of  $f$ . Note that  $W$  is holomorphic in  $1/\Lambda^{2N_c-3}$ , the inverse of the instanton factor.

Intuitively, one thinks of such exotics as being large and heavy bound states. Here we see that they become massless at  $\langle M \rangle = 0$ . This phenomenon is similar to the massless composite mesons and baryons found in  $SU(N_c)$  theories with  $N_f = N_c + 1$  [2]. Also, as with the  $N_f = N_c - 4$  theories discussed in sect. 3.2, we again see confinement without chiral symmetry breaking.

### 3.4. $N_f = N_c - 2$ ; the Coulomb phase

Since  $M^{ij}$  is neutral under the anomaly free  $U(1)_R$  symmetry, no superpotential can be generated; the theory has a quantum moduli space of physically inequivalent vacua labeled by the expectation values  $\langle M^{ij} \rangle$ . In this space of vacua the  $SO(N_c)$  gauge group is broken to  $SO(2) \cong U(1)$ . Hence the theory has a Coulomb phase with a massless photon supermultiplet. Classically, there is a singularity at  $\det M = 0$  associated with the larger unbroken gauge symmetry there. In the quantum theory, we find a different sort of singularity at  $\det M = 0$ , associated with massless monopoles rather than massless vector bosons.

The Coulomb phase of these theories can be explored by determining the effective gauge coupling  $\tau = \frac{\theta_{eff}}{\pi} + i \frac{8\pi}{g_{eff}^2}$  of the massless photon on the moduli space of degenerate vacua. By the  $SU(N_f)$  flavor symmetry,  $\tau$  depends on the vacuum  $\langle M^{ij} \rangle$  only via the  $SU(N_f)$  flavor singlet  $U \equiv \det M^{ij}$ . As in [5-7],  $\tau(U, \Lambda_{N_c, N_c-2})$  is naturally expressed in terms of a curve which can be exactly determined by holomorphy, the symmetries, and the requirement that  $\tau$  reproduces known behavior in various understood limits.

Consider the region of the moduli space where  $N_c - 4$  eigenvalues of  $M^{ij}$  are large. There the  $SO(N_c)$  theory is broken to a low-energy  $SO(4) \cong SU(2)_L \times SU(2)_R$  theory with  $N_f = 2$ . Matching the running gauge coupling at the Higgs scales, the low energy theory has dynamical scales given by  $\Lambda_{L,2}^4 = \Lambda_{R,2}^4 = \Lambda_{N_c, N_c-2}^{2N_c-4}/U_H$ , where  $U_H$  is the product of the  $N_c - 4$  large eigenvalues of  $M^{ij}$ . The flavor singlet combination of the light matter in the low energy  $SU(2)_L \times SU(2)_R$  theory is  $\hat{U} = U/U_H$ . In this limit, the curve should reproduce the one found in [7] for  $SU(2)_L \times SU(2)_R$  with  $N_f = 2$ :

$$y^2 = x^3 + x^2(-\hat{U} + 4\Lambda_L^4 + 4\Lambda_R^4) + 16\Lambda_L^4\Lambda_R^4x \quad (3.11)$$

(this is the curve of [7] upon normalizing  $M$  and  $\Lambda_{L,R}$  as in sect. 2.2). In terms of the original high-energy theory, (3.11) gives (upon rescaling  $x$  and  $y$ )

$$y^2 = x^3 + x^2(-U + 8\Lambda_{N_c, N_c-2}^{2N_c-4}) + 16\Lambda_{N_c, N_c-2}^{4N_c-8}x. \quad (3.12)$$

The exact curve must reproduce (3.12) in the limit where  $U$  is large compared to  $\Lambda_{N_c, N_c-2}^{2N_c-4}$ . Assuming as in [5,6] that the quantum corrections to (3.12) are polynomials in the instanton factor  $\Lambda_{N_c, N_c-2}^{2N_c-4}$ , holomorphy and the symmetries prohibit any corrections to (3.12). Hence, the curve (3.12) is exact.

The effective gauge coupling  $\tau$  obtained from (3.12) is singular at  $U = 0$  and at  $U = U_1 \equiv 16\Lambda_{N_c, N_c-2}^{2N_c-4}$ . It is found from (3.12) that, up to an overall conjugation by  $T^2$ , there is a monodromy  $\mathcal{M}_0 = S^{-1}TS$  in taking  $U \rightarrow e^{2\pi i}U$  around  $U = 0$  and a monodromy  $\mathcal{M}_1 = (ST^{-2})^{-1}TST^{-2}$  in taking  $U$  around  $U_1$ . This singular behavior reveals the presence of massless magnetic monopoles (or dyons) for vacua  $\langle M^{ij} \rangle = M^*$  with  $\det M^* = 0$  or  $\det M^* = U_1$ . Note that the spaces of such singular vacua  $M^*$  are non-compact.

The number of massless monopoles (or dyons) in a singular vacuum  $M^*$  follows from the monodromy of  $\tau$  upon taking  $M$  around  $M^*$ . We first consider vacua  $M^*$  with  $U = U_1$ . Taking  $M$  around such an  $M^*$ ,  $(M - M^*) \rightarrow e^{2\pi i}(M - M^*)$ , takes  $U - U_1 \rightarrow e^{2\pi i}(U - U_1)$

and thus gives the monodromy  $\mathcal{M}_1$ . This monodromy is associated with a single pair of monopoles (or dyons)  $E^\pm$ , of magnetic charge  $\pm 1$ , with a superpotential

$$W = (U - U_1) \left[ 1 + \mathcal{O} \left( \frac{U - U_1}{\Lambda_{N_c, N_c-2}^{2(N_c-2)}} \right) \right] E^+ E^-. \quad (3.13)$$

Away from  $U = U_1$  the monopoles are massive. At  $U = U_1$  they become massless and the photon gauge coupling is, therefore, singular.

The effective superpotential (3.13) properly describes the theory only in the vicinity of the moduli space of vacua with  $\det M = U_1$ , where the monopoles  $E^\pm$  are light. In particular, it is not valid in the vicinity of  $\det M = 0$ , where another set of monopoles are light. The spectrum of light monopoles near  $U = 0$  is more interesting than the single light monopole of (3.13) near  $U = U_1$ . As above, the light spectrum of monopoles follows from considering the monodromy implied by the curve (3.12) around a singular vacuum. Consider taking  $M$  around a vacuum  $M^*$  with  $\det M^* = 0$ ,  $(M - M^*) \rightarrow e^{2\pi i}(M - M^*)$ . This takes  $U \rightarrow e^{2\pi i(N_f - r)}U$ , where  $r$  is the rank of  $M^*$ , and thus gives the monodromy  $\mathcal{M}_0^{N_f - r}$ . Therefore, there must be  $N_f - r$  pairs of massless monopoles in a vacuum  $\langle M \rangle = M^*$  with  $M^*$  of rank  $r$ . This behavior corresponds to having  $N_f$  pairs of monopoles  $q_i^+$  and  $q_i^-$ , of magnetic charge  $\pm 1$ , with a superpotential

$$W = \frac{1}{2\mu} f \left( t = \frac{\det M}{\Lambda^{2(N_c-2)}} \right) M^{ij} q_i^+ q_j^-, \quad (3.14)$$

with  $f(t)$  holomorphic around  $t = 0$  and normalized so that  $f(0) = 1$ , to give rank  $(M)$  of the monopoles a mass. As in (3.8), the scale  $\mu$  was introduced because  $M$  has dimension two and  $q$  has dimension one. In order for the superpotential (3.14) to respect the global flavor symmetry, the monopoles  $q_i^\pm$  must have the  $SU(N_f) \times U(1)_R$  quantum numbers  $(\overline{N}_f)_1$ .

It follows from (3.12) that the massless magnetic particles at the singularities  $q_i^\pm$  and  $E^\pm$  have electric charges and global quantum numbers compatible with the identification  $q_i^\pm Q^i \sim E^\pm$ . The monopoles and dyons are visible in the semiclassical regime of large  $|U| = |\det M|$ . For  $M$  proportional to the identity matrix the global symmetry is broken to  $SO(N_f) \times U(1)_R$ . We expect to find states which are  $SO(N_f)$  singlets and states which are  $SO(N_f)$  vectors (and perhaps others). The electric charges of these states are determined up to an even integer associated with the monodromy  $U \rightarrow e^{2\pi i}U$  ( $\theta \rightarrow \theta + 2\pi$ ). By shifting  $\theta$  by  $\pi$  the electric charges are shifted by one unit. A more detailed analysis of

the semiclassical spectrum can determine all the quantum numbers of the states subject to some conventions; we did not perform such an analysis. In what follows we will refer to the massless particles at the origin,  $q_i^\pm$ , as magnetic monopoles and to the massless particles at  $\det M = 16\Lambda_{N_c, N_c-2}^{2N_c-4}$ ,  $E^\pm$ , as dyons.

The  $N = 1$  photon field strength  $\mathcal{W}_\alpha$  can be given a gauge invariant description on the moduli space in terms of the fundamental fields as

$$\mathcal{W}_\alpha \sim W_\alpha(Q)^{N_c-2} \quad (3.15)$$

where both the color and the flavor indices are contracted antisymmetrically. This relation will be generalized in the non-Abelian Coulomb phase discussed in the next section.

At the origin  $\langle M \rangle = 0$ , the monopoles  $q_i^\pm$  are all massless. The massless spectrum at the origin also consists of the fields  $M^{ij}$  and the photon supermultiplet. This spectrum, obtained from the monodromies of (3.12), satisfies two non-trivial consistency checks. First, at  $\langle M \rangle = 0$  the full  $SU(N_f) \times U(1)_R$  global symmetry is unbroken and the 't Hooft anomalies of this massless spectrum must match the anomalies (3.4) of the classical spectrum. The fermion components of the  $q_i^\pm$  and the photino give the combined anomalies

$$\begin{aligned} U(1)_R & 1 \\ U(1)_R^3 & 1 \\ SU(N_f)^3 & -2d_3(N_f) \\ SU(N_f)^2 U(1)_R & 0. \end{aligned} \quad (3.16)$$

Adding these to the contributions (3.5) of the field  $M$ , the anomalies do indeed match the microscopic anomalies (3.4) for  $N_f = N_c - 2$ .

Another check is to verify that, upon adding the term  $W_{tree} = \frac{1}{2}mM^{N_f N_f}$  and integrating out the field  $Q^{N_f}$ , we properly reproduce our description of  $N_f = N_c - 3$  discussed in the previous subsection. The equations of motion in the low energy effective theory with  $W_{tree}$  added lock the theory to be on a branch with  $\det M = U_1$  or on a branch with  $\det M = 0$ . On the branch with  $\det M = U_1$ , the equations of motion obtained upon adding  $W_{tree}$  to (3.13) give  $\langle E^+ E^- \rangle = -m/2 \det \widehat{M}$ , where  $\widehat{M}$  are the mesons for the remaining  $N_c - 3$  light flavors. The non-zero expectation values of  $\langle E^\pm \rangle$  lift the photon and confine electric charges. In fact, since  $E^\pm$  are dyons, this phenomenon is oblique confinement [22,23]. The remaining superpotential is

$$W = \frac{1}{2}mM^{N_f N_f} = 8 \frac{m\Lambda_{N_c, N_c-2}^{2N_c-4}}{\det \widehat{M}}. \quad (3.17)$$



Using the matching relation between the high energy scale  $\Lambda_{N_c, N_c-2}$  and the scale  $\Lambda_{N_c, N_c-3}$  of the low energy theory, (3.17) corresponds to the  $\epsilon = 1$  branch of (3.6).

Another branch is found by adding  $W_{tree} = \frac{1}{2}mM^{N_f N_f}$  to (3.14). The classical equations of motion show that  $\langle q_{N_f}^\pm \rangle \neq 0$  and hence the magnetic  $U(1)$  is Higgsed. This is confinement of the original electric variables. The non-trivial function  $f(t)$  in (3.14) and the constraint from the  $U(1)$  D-term make the explicit integration out of the massive modes complicated. However, it is easy to see that only  $\widehat{M}^{\hat{i}\hat{j}}$  with  $\hat{i}, \hat{j} = 1, \dots, N_f - 1$  and  $q_{\hat{i}} = \frac{1}{2\sqrt{m\mu}}(q_{\hat{i}}^+ q_{N_f}^- - q_{\hat{i}}^- q_{N_f}^+)$  remain massless. (This expression for  $q_{\hat{i}}$  is the gauge invariant interpolating field for the massless component of  $q_{\hat{i}}^\pm$ .) Their effective superpotential is

$$W = \frac{1}{2\mu} \hat{f} \left( \hat{t} = \frac{(\det \widehat{M})(\widehat{M}^{\hat{i}\hat{j}} q_{\hat{i}} q_{\hat{j}})}{m \Lambda_{N_c, N_c-2}^{2(N_c-2)}} \right) \widehat{M}^{\hat{i}\hat{j}} q_{\hat{i}} q_{\hat{j}} \quad (3.18)$$

where  $\hat{f}(\hat{t})$  depends on  $f(t)$  in (3.14). (The condition from the  $U(1)$  D-term is important in showing that a non-trivial  $f(t)$  leads to a non-trivial  $\hat{f}(\hat{t})$ ). This branch thus yields the  $\epsilon = -1$  branch of the low energy  $N_f = N_c - 3$  theory, as described by (3.8).

In the  $N = 2$  theory of [5] there are also massless monopoles and dyons which lead to confinement when they condense. In that case the confining branch and the oblique confinement branch are related by a global  $Z_2$  symmetry. Therefore, there is no physical difference between them. In some of the examples in [6] there are massless monopoles and dyons which are not related by any global symmetry. When they condense they lead to confinement and oblique confinement. However, since these theories have matter fields in the fundamental representation of the gauge group, there is no invariant distinction between Higgs, confinement and oblique confinement [21] in these examples. In the present cases the Higgs, confining and oblique confinement branches are physically inequivalent.

We see here an interesting physical phenomenon. Upon giving  $Q^{N_f}$  a mass, some of the magnetic monopoles  $q_i^\pm$  condense, leading to confinement, and the remaining massless monopoles are interpreted as massless exotics (or glueballs). A similar phenomenon was observed in  $SU(N_c)$  theories in [10] where massless magnetic quarks became massless baryons. We conclude that this phenomenon is generic; some of the gauge invariant composites (baryons, glueballs, exotics) can be thought of as “magnetic.”

#### 4. $N_c \geq 4$ , $N_f \geq N_c - 1$ ; magnetic $SO(N_f - N_c + 4)$ gauge theory

As discussed in [10], the infra-red behavior of these theories has a dual, magnetic description in terms of an  $SO(N_f - N_c + 4)$  gauge theory with  $N_f$  flavors of dual quarks  $q_i$  and the additional gauge singlet field  $M^{ij}$ . For  $N_c - 2 < N_f \leq \frac{3}{2}(N_c - 2)$  the magnetic degrees of freedom are free in the infra-red while for  $\frac{3}{2}(N_c - 2) < N_f < 3(N_c - 2)$  the electric and the magnetic theories flow to the same non-trivial fixed point of the renormalization group. Although the two theories are different away from the extreme infra-red, they are completely equivalent at long distance. This means that the two (super) conformal field theories at long distance are identical, having the same correlation functions of all of the operators, including high dimension (irrelevant) operators.

The fields in the magnetic theory have the anomaly free  $SU(N_f) \times U(1)_R$  charges

$$\begin{aligned} q & \quad (\overline{N_f})_{\frac{N_c-2}{N_f}} \\ M & \quad \left(\frac{1}{2}N_f(N_f+1)\right)_2 \frac{(N_f-N_c+2)}{N_f} \end{aligned} \quad (4.1)$$

and a superpotential

$$W = \frac{1}{2\mu} M^{ij} q_i \cdot q_j \quad (4.2)$$

(an additional term is required for  $N_f = N_c - 1$ ). The scale  $\mu$  is needed for the following reason. In the electric description  $M^{ij} = Q^i \cdot Q^j$  has dimension two at the UV fixed point and acquires some anomalous dimension at the IR fixed point. In the magnetic description  $M$  is an elementary field of dimension one at the UV fixed point. Denote it by  $M_m$ . In order to relate it to  $M$  of the electric description a scale  $\mu$  must be introduced with the relation  $M = \mu M_m$ . Below we will write all the expressions in terms of  $M$  and  $\mu$  rather than in terms of  $M_m$ .

For generic  $N_c$  and  $N_f$  the scale of the magnetic theory,  $\tilde{\Lambda}$ , is related to that of the electric theory,  $\Lambda$ , by

$$\Lambda^{3(N_c-2)-N_f} \tilde{\Lambda}^{3(N_f-N_c+2)-N_f} = C(-1)^{N_f-N_c} \mu^{N_f} \quad (4.3)$$

where  $C$  is a dimensionless constant which we will determine below and  $\mu$  is the dimensional scale explained above. This relation of the scales has several consequences:

1. It is easy to check that it is preserved under mass deformations and along the flat directions (more details will be given below). The phase  $(-1)^{N_f-N_c}$  is important in order to ensure that this is the case.

2. It shows that as the electric theory becomes stronger the magnetic theory becomes weaker and vice versa.

3. Because of the phase  $(-1)^{N_f - N_c}$ , the relation (4.3) does not look dual – if we perform another duality transformation it becomes  $\Lambda^{3(N_c - 2) - N_f} \tilde{\Lambda}^{3(N_f - N_c + 2) - N_f} = C(-1)^{N_c} \tilde{\mu}^{N_f}$  and therefore

$$\tilde{\mu} = -\mu. \quad (4.4)$$

This minus sign is important when we dualize again. The dual of the dual magnetic theory is an  $SO(N_c)$  theory with scale  $\Lambda$ , quarks  $d^i$ , and additional singlets  $M^{ij}$  and  $N_{ij} = q_i \cdot q_j$ , with superpotential

$$W = \frac{1}{2\tilde{\mu}} N_{ij} d^i \cdot d^j + \frac{1}{2\mu} M^{ij} N_{ij} = \frac{1}{2\mu} N_{ij} (M^{ij} - d^i \cdot d^j). \quad (4.5)$$

The first term is our standard superpotential of duality transformations (as pointed out in [10] the relative minus sign between it and (4.2), which follows from (4.4), is common in Fourier or Legendre transforms). The second term is simply copied from (4.2).  $M$  and  $N$  are massive and can be integrated out using their equations of motion  $N = 0$ ,  $M^{ij} = d^i \cdot d^j$ . This last relation shows that the quarks  $d$  can be identified with the original electric quarks  $Q$ . The dual of the magnetic theory is the original electric theory.

4. Differentiating the action with respect to  $\log \Lambda$  relates the field strengths of the electric and the magnetic theories as  $W_\alpha^2 = -\tilde{W}_\alpha^2$ . The minus sign in this expression is common in electric magnetic duality, which maps  $E^2 - B^2 = -(\tilde{E}^2 - \tilde{B}^2)$ . In our case it shows that the gluino bilinear in the electric and the magnetic theories are related by  $\lambda\lambda = -\tilde{\lambda}\tilde{\lambda}$ .

The electric theories are also invariant under a discrete  $Z_{2N_f}$  symmetry generated by  $Q \rightarrow e^{\frac{2\pi i}{2N_f}} Q$  and charge conjugation  $\mathcal{C}$ . In the dual description these are generated by  $q \rightarrow e^{\frac{-2\pi i}{2N_f}} \mathcal{C}q$  and  $\mathcal{C}$  respectively. Note that the  $Z_{2N_f}$  symmetry commutes with the electric gauge group but does not commute with the magnetic one. This is similar to the action of the parity operator  $P$  in Maxwell theory: in the dual description  $P$  is replaced with  $P\mathcal{C}$ .

The gauge invariant (primary) chiral operators of the electric theory are

$$\begin{aligned} M^{ij} &= \frac{1}{2} Q^i Q^j \\ B^{[i_1, \dots, i_{N_c}]} &= Q^{i_1} \dots Q^{i_{N_c}} \\ b^{[i_1, \dots, i_{N_c-4}]} &= W_\alpha^2 Q^{i_1} \dots Q^{i_{N_c-4}} \\ \mathcal{W}_\alpha^{[i_1, \dots, i_{N_c-2}]} &= W_\alpha Q^{i_1} \dots Q^{i_{N_c-2}} \end{aligned} \quad (4.6)$$

with the gauge indices implicit and contracted. These operators get mapped to gauge invariant operators of the magnetic theory as

$$\begin{aligned}
M^{ij} &\rightarrow M^{ij} \\
B^{[i_1 \dots i_{N_c}]} &\rightarrow \epsilon^{i_1 \dots i_{N_f}} \widetilde{b}_{i_{N_c+1} \dots i_{N_f}} \\
b^{[i_1 \dots i_{N_c-4}]} &\rightarrow \epsilon^{i_1 \dots i_{N_f}} \widetilde{B}_{[i_{N_c-3} \dots i_{N_f}]} \\
\mathcal{W}_\alpha^{[i_1, \dots, i_{N_c-2}]} &\rightarrow \epsilon^{i_1 \dots i_{N_f}} (\widetilde{\mathcal{W}}_\alpha)_{[i_{N_c-1}, \dots, i_{N_f}]},
\end{aligned} \tag{4.7}$$

where  $\widetilde{B}$ ,  $\widetilde{b}$ , and  $\widetilde{\mathcal{W}}_\alpha$  are the magnetic analogs of the operators in (4.6). The last of these relations has already been noted in (3.15). Note that these maps are consistent with both the continuous and the discrete symmetries.

#### 4.1. $N_f = N_c - 1$ ; magnetic $SO(3)$ gauge theory

The dual magnetic description is in terms of an  $SO(3)$  gauge theory with  $SO(3)$  quarks  $q_i$  in  $(\overline{N}_f)_{\frac{N_c-2}{N_f}}$  of the global  $SU(N_c) \times U(1)_R$ ,  $SO(3)$  singlets  $M^{ij}$  with  $SU(N_c) \times U(1)_R$  quantum numbers as before, and a superpotential

$$W = \frac{1}{2\mu} M^{ij} q_i \cdot q_j - \frac{1}{64\Lambda_{N_c, N_c-1}^{2N_c-5}} \det M, \tag{4.8}$$

where  $\mu$  is the dimensionful normalization factor discussed in the introduction to this section. The scale  $\widetilde{\Lambda}_{3, N_c-1}$  of the magnetic  $SO(3)$  theory with  $N_f = N_c - 1$  massless quarks is related to the scale of the electric theory by

$$2^{14} (\Lambda_{N_c, N_c-1}^{2N_c-5})^2 \widetilde{\Lambda}_{3, N_c-1}^{6-2(N_c-1)} = \mu^{2(N_c-1)}. \tag{4.9}$$

Since this relation is like the square of (4.3), the phase discussed there is not present. The normalizations of the second term in (4.8) and of the relation (4.9) are determined for consistency of the various deformations of the theory (see below). Since  $N_f \geq 3$ , the magnetic  $SO(3)$  gauge theory is not asymptotically free and is, therefore, free in the infra-red. At the infra-red fixed point the free fields  $q_i$  and  $M$  all have dimension one. Hence, the superpotential (4.8) is irrelevant and the infra-red theory has a large accidental symmetry. However the superpotential (4.8), including the  $\det M$  term, is essential in order to properly describe the theory when perturbed by mass terms or along flat directions. Also, without the  $\det M$  term, the magnetic theory (4.8) would have a  $Z_{4N_f}$  symmetry in contrast to the  $Z_{2N_f}$  symmetry (2.4) of the electric theory. Using the symmetries and holomorphy around

$M = q = 0$ , it is easy to see that, unlike (3.8) or (3.14), (4.8) cannot be modified by a non-trivial function of the invariants.

At the origin  $\langle M \rangle = 0$ , the fields  $M^{ij}$ ,  $q_i$  and the  $SO(3)$  vector bosons are all massless. The  $SU(N_c) \times U(1)_R$  anomalies associated with this massless spectrum match the anomalies (3.4) of the classical spectrum [10].

### Flat directions

We now consider the moduli space of vacua in the dual magnetic description, verifying that it agrees with the moduli space of vacua in the original electric description. For  $M \neq 0$ , (4.8) gives the magnetic quarks a mass matrix  $\mu^{-1}M$ . The low energy theory is the magnetic  $SO(3)$  with  $k = N_f - \text{rank}(M)$  massless dual quarks  $q$ . For  $\text{rank}(M) = N_f$  all the dual quarks are massive. Then, using (2.5) in the magnetic theory, the low energy, pure gauge,  $SO(3)$  Yang-Mills theory has a scale  $\tilde{\Lambda}_{3,0}^6 = \tilde{\Lambda}_{3,N_f}^{6-2(N_c-1)} \det(\mu^{-1}M)^2$ . Gluino condensation in the magnetic  $SO(3)$  leads to two vacua,  $\langle \tilde{W}_\alpha^2 \rangle = \epsilon \tilde{\Lambda}_{3,0}^3$  with  $\epsilon = \pm 1$ , and hence an additive term  $2\langle \tilde{W}_\alpha^2 \rangle = 2\epsilon \tilde{\Lambda}_{3,N_f}^{6-2(N_c-1)} \mu^{-N_c+1} \det M$  in the superpotential. Adding this to the term proportional to  $\det M$  in (4.8) and using (4.9), the  $\epsilon = 1$  branch reproduces the moduli space of supersymmetric ground states with generic  $\langle M \rangle$ . The only massless fields on this moduli space of generic  $\langle M \rangle$  are the components of  $M$ . The  $\epsilon = -1$  branch will be interpreted in sect. 6.

For  $\text{rank}(M) = N_f - 1$  the low energy theory is the magnetic  $SO(3)$  with one massless flavor, which we take to be  $q_{N_f}$ . This is a magnetic version of the theory analyzed in [5]. It has a massless photon and massless monopoles at  $\langle u \rangle \equiv \langle q_{N_f}^2 \rangle = 4\epsilon \tilde{\Lambda}_{3,1}^2$  ( $\epsilon = \pm 1$ ) where, using (4.9) and (2.5), the scale  $\tilde{\Lambda}_{3,1}$  of the low energy theory is given by  $\tilde{\Lambda}_{3,1}^4 = 2^{-14} (\mu \Lambda^{5-2N_c} \det \hat{M})^2$ , with  $\det \hat{M} = \det M / M^{N_f N_f}$  the product of the  $N_f - 1$  non-zero eigenvalues of  $M$ . The low energy superpotential near the massless monopole points  $u \approx 4\epsilon \tilde{\Lambda}_{3,1}^2$  is

$$W = \frac{1}{2\mu} M^{N_f N_f} \left( u - \frac{1}{32 \Lambda_{N_c, N_c-1}^{2N_c-5}} \mu \det \hat{M} \right) - \frac{1}{2\mu} (u - 4\epsilon \tilde{\Lambda}_{3,1}^2) \tilde{E}_{(\epsilon)}^+ \tilde{E}_{(\epsilon)}^- \quad (4.10)$$

where the normalization of the  $\tilde{E}_{(\epsilon)}^+ \tilde{E}_{(\epsilon)}^-$  term was arranged for convenience. The  $M^{N_f N_f}$  equations of motion give  $\langle u \rangle = 2^{-5} \Lambda_{N_c, N_c-1}^{5-2N_c} \mu \det \hat{M}$ , fixing the magnetic theory to the  $\epsilon = +1$  supersymmetric ground state with a massless monopole  $\tilde{E}_{(+)}^\pm$  in addition to the massless photon. The  $u$  equation of motion gives  $M^{N_f N_f} = \tilde{E}_{(+)}^+ \tilde{E}_{(+)}^-$ . The monopole  $\tilde{E}_{(+)}^\pm$  is magnetic relative to the magnetic  $SO(3)$  variables; it is electric in terms of the original

electric variables. Indeed, using the electric theory we easily see that a flat direction with rank  $(M) = N_f - 1 = N_c - 2$  breaks the  $SO(N_c)$  gauge group to  $SO(2) \cong U(1)$  with one of the elementary quarks which is charged under this  $U(1)$  remaining massless. In the magnetic description we find it as a massless collective excitation. This interpretation is strengthened by the relation  $M^{N_f N_f} = \tilde{E}_{(+)}^+ \tilde{E}_{(+)}^-$ .

For rank  $(M) \leq N_f - 2$  the low energy theory is  $SO(3)$  with  $k = N_f - \text{rank}(M) \geq 2$  flavors which is either free or is at a non-trivial fixed point (only for  $k = 2$ ) of the beta function. It is dual to the answer one gets in the electric variables.

### Mass deformations

Adding a  $Q^{N_f}$  mass term  $W_{tree} = \frac{1}{2}mM^{N_f N_f}$  to the magnetic theory (4.8), the  $M^{N_f N_f}$  equation of motion gives  $q_{N_f}^2 = 2^{-5}\mu\Lambda_{N_c, N_c-1}^{5-2N_c} \det \widehat{M} - \mu m$ ; this generically breaks the magnetic  $SO(3)$  gauge group to  $SO(2)$ . Integrating out the massive fields, the low energy magnetic  $SO(2)$  theory has neutral fields  $M^{\hat{i}\hat{j}}$  and fields  $q_i^\pm$  of  $SO(2)$  charge  $\pm 1$ , where  $\hat{i} = 1 \dots N_c - 2$ , along with a superpotential  $W_{tree} = \frac{1}{2\mu}M^{\hat{i}\hat{j}}q_i^+q_j^-$ . Instantons in the broken magnetic  $SO(3)$  can generate additional terms<sup>4</sup>, modifying the superpotential to  $W = \frac{1}{2\mu}f(\det \widehat{M}/\Lambda_{N_c, N_c-2}^{2(N_c-2)})M^{\hat{i}\hat{j}}q_i^+q_j^-$ . This is the theory (3.14), with the  $q_i^\pm$  becoming the monopoles of the theory with  $N_f = N_c - 2$ . Therefore, the  $SO(3)$  gauge group of (4.8) really deserves to be called “magnetic.”

There is another special point on the moduli space of vacua. For large  $\det \widehat{M}$  the first  $N_f - 1$  of the  $q_i$  should be integrated out and the low energy theory is the magnetic  $SO(3)$  theory with the quark  $q_{N_f}$  and scale  $\tilde{\Lambda}_{3,1}^4 = 2^{-14}(\mu\Lambda_{N_c, N_c-1}^{5-2N_c} \det \widehat{M})^2$ . The term  $\frac{1}{2}M^{N_f N_f}u$  in the superpotential, where  $u = q_{N_f}^2$ , locks the magnetic theory to be at one of the vacua where the theory has massless monopoles. Near these two vacua the low energy theory is described by

$$W \approx \frac{1}{2\mu}M^{N_f N_f}(u - \frac{1}{32\Lambda_{N_c, N_c-1}^{2N_c-5}}\mu \det \widehat{M} + \mu m) - \frac{1}{2\mu}(u - 4\epsilon\tilde{\Lambda}_{3,1}^2)\tilde{E}_{(\epsilon)}^+\tilde{E}_{(\epsilon)}^-, \quad (4.11)$$

where the fields  $\tilde{E}_{(\epsilon)}^\pm$ , which appear as the result of strong coupling phenomena in the magnetic theory, can be interpreted as a monopole or dyon of that theory. Using the

---

<sup>4</sup> As in the discussion of footnote 2, these terms should be included because, when the magnetic  $SO(3)$  is broken to  $SO(2)$ , there are well-defined instantons in the broken part of the gauge group.

equations of motion, there is no supersymmetric vacuum for  $\epsilon = 1$  and the low energy theory for the  $\epsilon = -1$  vacuum is

$$W \approx \frac{1}{2} \left( m - \frac{1}{16\Lambda_{N_c, N_c-1}^{2(N_c-2)-1}} \det \widehat{M} \right) \widetilde{E}_{(-)}^+ \widetilde{E}_{(-)}^-. \quad (4.12)$$

This theory corresponds to (3.13), with  $\widetilde{E}_{(-)}^\pm$  the dyons which are massless at  $\det \widehat{M} = 16\Lambda_{N_c, N_c-2}^{2N_c-4}$ .

Adding more masses, we gradually reduce  $N_f$ . The monopoles or dyons condense and lead to confinement or oblique confinement. For fewer than  $N_c - 4$  massless flavors the confining branch disappears and there is only the oblique confinement branch with

$$W_{oblique} = -\frac{1}{32\Lambda_{N_c, N_c-1}^{2N_c-5}} \det M, \quad (4.13)$$

which is the continuation of (3.1) to  $N_f = N_c - 1$ . The superpotential (4.13) does not mean that the flat directions of the massless theory are lifted. As in [7], it should be only used to reproduce  $\langle M \rangle$  when the quarks are massive. It is present only in the oblique confinement branch and not in the Higgs branch.

To conclude, the monopoles  $q_i^\pm$  at the origin of the  $N_f = N_c - 2$  theories have a weakly coupled magnetic description in terms of the components  $q_i^\pm$  of the quarks in the dual theory. The massless dyons  $E^\pm$  at  $\det \widehat{M} = 16\Lambda_{N_c, N_c-2}^{2N_c-4}$  appear strongly coupled both in the original electric description and in terms of the dual magnetic description.

#### 4.2. $N_f = N_c$ ; magnetic $SO(4)$ gauge theory

The electric  $SO(N_c)$  theory with  $N_f = N_c$  quarks has the gauge invariant “baryon” operator  $B = \det Q$  in addition to the “mesons”  $M^{ij} = Q^i \cdot Q^j$ . As discussed in sect. 2, the classical moduli space of vacua is constrained by  $B = \pm \sqrt{\det M}$ .

As discussed in [10], these theories have a dual magnetic description in terms of an  $SO(4) \cong SU(2)_L \times SU(2)_R$  gauge theory with  $N_f$  flavors of quarks  $q_i$  in the  $(2, 2)$  dimensional representation of  $SU(2)_L \times SU(2)_R$  along with the  $SO(4)$  singlets  $M^{ij}$  and the superpotential

$$W = \frac{1}{2\mu} M^{ij} q_i \cdot q_j. \quad (4.14)$$

As in the previous subsection, the symmetries and holomorphy around  $M = q = 0$  uniquely determine this superpotential; unlike (3.8) or (3.14), (4.14) cannot be modified by a non-trivial function.

The scales  $\tilde{\Lambda}_{s,N_c}$  of the magnetic  $SU(2)_s$  are equal and are related to the electric scale by

$$2^8 \tilde{\Lambda}_{s,N_c}^{6-N_c} \Lambda_{N_c,N_c}^{2N_c-6} = \mu^{N_c} \quad \text{for } s = L, R. \quad (4.15)$$

The magnetic  $SU(2)_L \times SU(2)_R$  theory is not asymptotically free for  $N_f > 5$ . Therefore, for  $N_f > 5$  the magnetic theory is free in the infra-red. For  $N_c = N_f = 4, 5$  the theory is asymptotically free and has an interacting fixed point with the quarks  $q_i$  and the  $SU(2)_L \times SU(2)_R$  gauge fields in an interacting non-Abelian Coulomb phase. It is dual to the electric description in terms of the original  $SO(N_c)$  theory with  $N_f = N_c$  quarks  $Q^i$  in a non-Abelian Coulomb phase.

The theory has an anomaly free  $SU(N_f) \times U(1)_R$  global symmetry with the fields  $q_i$  in the  $(\overline{N}_f)_{\frac{N_c-2}{N_c}}$  and the  $M^{ij}$  in the  $(\frac{1}{2}N_f(N_f+1))_{\frac{4}{N_c}}$ . At  $\langle M \rangle = 0$  the fields  $q_i$  and  $M^{ij}$  are all massless. Since the global  $SU(N_f) \times U(1)_R$  symmetry is unbroken at the origin, the 't Hooft anomalies of this massless spectrum must match the classical anomalies (3.4); they do indeed match [10].

#### Flat directions

Consider the theory in a vacuum of non-zero  $\langle M \rangle$ . The  $M$  equations of motion give  $q_i \cdot q_j = 0$ ; the  $SO(4)$  D-terms imply that the only solution is  $\langle q_i \rangle = 0$ . The low energy theory around this point is the magnetic  $SO(4)$  with  $k = N_f - \text{rank}(M)$  dual quarks  $q$ . For  $\text{rank}(M) = N_f$  there are no light dual quarks and the low-energy magnetic theory is  $SU(2)_L \times SU(2)_R$  Yang-Mills theory with, using (2.5) and (4.15), the scales  $\tilde{\Lambda}_{s,0}^6 = 2^{-8} \det M \Lambda_{N_c,N_c}^{6-2N_c}$  for  $s = L, R$ . There is gaugino condensation in the  $SU(2)_s$ :  $\langle (\widetilde{W}_\alpha)_s^2 \rangle = \epsilon_s \tilde{\Lambda}_{s,0}^3$  for  $s = L, R$ ;  $\epsilon_s = \pm 1$  label four vacua. This leads to a superpotential  $W = 2(\epsilon_L + \epsilon_R) \tilde{\Lambda}^3$ . The two vacua with  $\epsilon_L \epsilon_R = 1$  have  $W \approx \pm \frac{1}{4} \Lambda^{3-N_f} (\det M)^{1/2}$  and do not lead to supersymmetric vacua. The two vacua with  $\epsilon_L \epsilon_R = -1$  give a branch with  $W = 0$ ; each gives a supersymmetric ground state. We thus find that there are two vacua for  $\text{rank}(M) = N_f$ , corresponding to the sign of  $\langle (\widetilde{W}_\alpha)_L^2 - (\widetilde{W}_\alpha)_R^2 \rangle$ . This is in agreement with the classical moduli space of vacua discussed in sect. 2 with, by the identification [10]  $B \sim (\widetilde{W}_\alpha)_L^2 - (\widetilde{W}_\alpha)_R^2$ , the two vacua for  $\langle M \rangle$  of rank  $N_c$  corresponding to the sign of  $B = \pm \sqrt{\det M}$ .

For  $\text{rank}(M) = N_f - 1$ , the low energy theory is the magnetic  $SO(4)$  theory with one flavor,  $q_{N_f}$ . As discussed in sect. 3.3, it has no massless gauge fields and a massless composite  $\tilde{q}$  appears. In this case, the massless composite is a glueball  $(\widetilde{W}_\alpha)_L^2 - (\widetilde{W}_\alpha)_R^2$ . The effective Lagrangian is  $W = \frac{1}{2\mu} N_{N_f N_f} (M^{N_f N_f} - \tilde{q}^2)$ ; integrating out  $N^{N_f N_f} = q_{N_f} \cdot q_{N_f}$



gives  $M^{N_f N_f} = \tilde{q}^2$ . The metric is smooth in terms of  $\tilde{q}$  rather than  $M^{N_f N_f}$ . The field  $\tilde{q}$  of the magnetic theory can be seen semiclassically in the electric theory. For rank  $(M) = N_f - 1 = N_c - 1$  the electric theory is completely Higgsed but one of the quarks remains massless. Its gauge invariant interpolating field is  $B = \det Q$ , which is indeed mapped under the duality to the massless glueball  $(\widetilde{W}_\alpha)_L^2 - (\widetilde{W}_\alpha)_R^2$  of the magnetic theory.

For rank  $(M) = N_f - 2$  the low energy theory is the magnetic  $SO(4)$  with two flavors discussed in sect. 3.4, which is in the Coulomb phase with massless magnetic monopoles. These can be seen in the electric theory as being some of the components of the elementary quarks, which are charged under the unbroken electric  $U(1)$  for rank  $(M) = N_f - 2 = N_c - 2$ .

For rank  $(M) = N_f - 3$  the low energy theory is the magnetic  $SO(4)$  with three flavors discussed in sect. 4.1. It is in a free non-Abelian magnetic phase with gauge group  $SO(3)$  with three flavors of magnetic quarks. These are precisely the electric degrees of freedom of the underlying  $SO(N_c)$  theory, which is Higgsed along the flat directions with rank  $(M) = N_f - 3$  to an electric  $SO(3)$  subgroup. Here we see these elementary quarks and gluons appearing out of strong coupling dynamics in the dual magnetic theory.

For rank  $(M) \leq N_f - 4$  the low energy theory is the magnetic  $SO(4)$  with more than three flavors. It is either at a non-trivial fixed point of the beta function or not asymptotically free.

We see that for rank  $(M) < N_f$  there is a unique ground state which can be interpreted either in the electric or in the magnetic theory.

### *Mass deformations*

Now consider perturbing the theory by  $W_{tree} = \frac{1}{2}mM^{N_c N_c}$  to give a mass to the  $N_c$ -th electric quark. Adding  $W_{tree}$  to (4.14), the  $M^{N_c N_c}$  equation of motion gives  $\langle q_{N_c}^2 \rangle = -\mu m$ , which breaks the magnetic  $SU(2)_L \times SU(2)_R$  to the diagonal  $SU(2)_D$ . The  $q_{N_c}$  equations of motion give  $M^{i N_c} = 0$  and the  $M^{\hat{i} N_c}$  equations of motion give  $q_{\hat{i}} \cdot q_{N_c} = 0$  for  $\hat{i} = 1 \dots N_c - 1$ . The remaining low energy theory is the diagonal magnetic  $SO(3)$  gauge theory with  $N_c - 1$  triplets  $\hat{q}_{\hat{i}}$  and the  $SO(3)$  singlets  $M^{\hat{i}\hat{j}}$ , where  $\hat{i}, \hat{j} = 1 \dots N_c - 1$ . These fields have a superpotential coming from (4.14). In addition, there is a contribution to the superpotential associated with instantons in the broken magnetic  $SU(2)_L$  and  $SU(2)_R$ . In particular, for  $\det \widehat{M} \neq 0$ , the superpotential (4.14) gives masses to the first  $N_c - 1$  dual quarks  $q_{\hat{i}}$ . The low energy theory has one dual quark  $q_{N_c}$  and the  $SU(2)_s$  scales are  $\tilde{\Lambda}_{s,1}^5 =$

$2^{-8}\mu \det \widehat{M} \Lambda_{N_c, N_c}^{6-2N_c}$ . Instantons in the broken magnetic  $SU(2)_s$  generate the superpotential  $W_{inst} = 2(\widetilde{\Lambda}_{L,1}^5 + \widetilde{\Lambda}_{R,1}^5)/q_{N_c} \cdot q_{N_c}$ . Using  $\langle q_{N_c} \cdot q_{N_c} \rangle = -m\mu$  and  $m\Lambda_{N_c, N_c}^{2N_c-6} = \Lambda_{N_c, N_c-1}^{2N_c-5}$ ,

$$W_{inst} = -\frac{1}{64} \Lambda_{N_c, N_c-1}^{5-2N_c} \det \widehat{M}. \quad (4.16)$$

Combining  $W_{inst}$  with the superpotential coming from (4.14), the low energy magnetic theory properly yields the magnetic  $SO(3)$  theory with  $N_c-1$  flavors and the superpotential (4.8) discussed in the previous subsection. Using (2.5) and (2.6) in the electric and magnetic theories, the scale relation (4.15) properly yields the scale relation (4.9) for the low-energy electric and magnetic theories.

#### 4.3. $N_f > N_c$ ; magnetic $SO(N_c - N_c + 4)$ gauge theory

The superpotential in the dual magnetic description is

$$W = \frac{1}{2\mu} M^{ij} q_i \cdot q_j. \quad (4.17)$$

As in the previous subsections, symmetries and holomorphy around  $M = q = 0$  uniquely determine this superpotential; a non-trivial function as in (3.8) or (3.14) cannot be present. The scale of the magnetic theory is related to that of the electric theory by

$$2^8 \Lambda_{N_c, N_f}^{3(N_c-2)-N_f} \widetilde{\Lambda}_{N_f-N_c+4, N_f}^{3(N_f-N_c+2)-N_f} = (-1)^{N_f-N_c} \mu^{N_f}. \quad (4.18)$$

The one-loop beta function of  $SO(N_f - N_c + 4)$  with  $N_f$  quarks reveals that the magnetic theory is not asymptotically free for  $N_f \leq \frac{3}{2}(N_c - 2)$ . For this range of  $N_f$ , the magnetic gauge theory is free in the infra-red and provides a weakly coupled description of the strongly coupled electric theory. For  $\frac{3}{2}(N_c - 2) < N_f < 3(N_c - 2)$ , the magnetic theory is asymptotically free and has an interacting fixed point with the  $SO(N_f - N_c + 4)$  theory in a non-Abelian Coulomb phase. For this range of  $N_f$ , there is also an electric description in terms of the original  $SO(N_c)$  theory with  $N_f$  quarks in a non-Abelian Coulomb phase. The magnetic description is at stronger coupling as  $N_f$  is increased and the electric description is at weaker coupling. For  $N_f \geq 3(N_c - 2)$ , the magnetic description is at infinite coupling whereas the electric description is free in the infra-red.

At the origin of  $\langle M \rangle$  the fields  $M^{ij}$ , the  $q_i$ , and the  $SO(N_f - N_c + 4)$  vector bosons are all massless. The 't Hooft anomalies of this massless spectrum match the anomalies (3.4) of the classical theory [10].

### *Flat directions*

Now consider the theory along the flat directions of non-zero  $\langle M \rangle$ .  $k = N_f - \text{rank}(M)$  of the  $q_i$  remain massless. The  $F$  and  $D$  terms of the dual theory fix  $\langle q_i \rangle = 0$ . For  $\text{rank}(M) > N_c$ , there is no supersymmetric ground state at  $\langle q_i \rangle = 0$  because a superpotential analogous to (3.1) is generated in the magnetic theory. For  $\text{rank}(M) = N_c$ , the low energy magnetic theory is analogous to the theory considered in sect. 3.2; there are two supersymmetric ground states at the origin corresponding to the two sign choices for  $\epsilon_L$  in the  $\epsilon_L \epsilon_R = -1$  branch of the magnetic analog of (3.3). The same is also true in the underlying electric theory. For  $\text{rank}(M) = N_c - 1$  the low energy theory is  $SO(N_f - N_c + 4)$  with  $k = N_f - N_c + 1$ . It is analogous to the theory considered in sect. 3.3. It has no massless gauge fields but massless composites. These can be interpreted as some of the components of the elementary electric quarks. For  $\text{rank}(M) = N_c - 2$  the low energy magnetic theory is  $SO(N_f - N_c + 4)$  with  $N_f - N_c + 2$  massless flavors. It is analogous to the theory discussed in sect. 3.4. This magnetic theory has a massless photon which is at infinite coupling because of the massless magnetic monopoles at the origin,  $\langle q_i \rangle = 0$ . This gives the dual description of the fact which is obvious in the electric variables: that there is a massless photon with massless charged elementary quarks when  $\text{rank}(M) = N_c - 2$ . For  $\frac{3}{2}N_c - \frac{1}{2}N_f - 3 \leq \text{rank}(M) < N_c - 2$  the low energy theory is still strongly coupled. It is dualized as in this section to a free electric theory  $SO(N_c - \text{rank}(M))$  with  $N_f - \text{rank}(M)$  massless quarks. Again, this result is obvious in the original electric variables. For  $\text{rank}(M) < \frac{3}{2}N_c - \frac{1}{2}N_f - 3$  the magnetic degrees of freedom are either interacting or free.

To summarize, the moduli space of supersymmetric vacua is given by the space of  $\langle M \rangle$  of rank at most  $N_c$  along with an additional sign when  $M$  is of rank  $N_c$ . We thus recover the classical moduli space, discussed in sect. 2, of the electric theory in terms of strong coupling effects in the magnetic description. Conversely, some of the strong coupling phenomena of the previous subsections can be understood from the classical moduli space of the dual magnetic theory.

### *Mass deformations*

Adding a  $Q^{N_f}$  mass term,  $W_{tree} = \frac{1}{2}mM^{N_f N_f}$ , to the electric theory gives a low energy electric  $SO(N_c)$  theory with  $N_f - 1$  massless quarks. Adding  $W_{tree}$  to the theory (4.17), the  $M^{N_f N_f}$  equations of motion give  $\langle q_{N_f} \rangle \neq 0$ , breaking the magnetic  $SO(N_f - N_c + 4)$  gauge theory with  $N_f$  quarks to  $SO(N_f - N_c + 3)$  with  $N_f - 1$  quarks. The  $q_{N_f}$  equations of motion

give  $M^{\hat{i}N_f} = 0$  and the  $M^{\hat{i}N_f}$  equations of motion give  $q_{\hat{i}} \cdot q_{N_f} = 0$  for  $\hat{i} = 1 \dots N_f - 1$ . The remaining low energy theory is then a magnetic  $SO(N_f - N_c + 3)$  theory with  $N_f - 1$  flavors and the superpotential (4.17). This low energy magnetic theory is, indeed, the magnetic dual to the low energy  $SO(N_c)$  theory with  $N_f - 1$  massless quarks. Using (2.5) and (2.6) in the electric and magnetic theories, the relation (4.18) in the high-energy theory implies that the scales of the low-energy theory are also related as in (4.18). When we flow from an electric theory with  $N_f = N_c + 1$  to the electric theory with  $N_f = N_c$ , the magnetic  $SO(5)$  with  $N_f = N_c + 1$  is broken to the magnetic  $SU(2)_L \times SU(2)_R$  of the previous section with  $N_f = N_c$ .

Another way to analyze the theory with mass terms is to consider the massless theory for generic values of  $M$ . The dual quarks acquire mass  $\frac{1}{\mu}M$  and the low energy magnetic theory is a pure gauge  $SO(N_f - N_c + 4)$  with scale  $\tilde{\Lambda}_L^{3(N_f - N_c + 2)} = \mu^{-N_f} \tilde{\Lambda}_{N_f - N_c + 4, N_f}^{3(N_f - N_c + 2) - N_f} \det M$ . Gluino condensation in this theory leads to an effective superpotential

$$\begin{aligned}
W_{eff} &= \frac{1}{2}(N_f - N_c + 2)2^{4/(N_f - N_c + 2)}\tilde{\Lambda}_L^3 \\
&= \frac{1}{2}(N_f - N_c + 2)\left(16\mu^{-N_f}\tilde{\Lambda}_{N_f - N_c + 4, N_f}^{3(N_f - N_c + 2) - N_f} \det M\right)^{1/(N_f - N_c + 2)} \\
&= \frac{1}{2}(N_f - N_c + 2)\left((-1)^{N_f - N_c} \frac{16\Lambda_{N_c, N_f}^{3N_c - 6 - N_f}}{\det M}\right)^{-1/(N_f - N_c + 2)} \\
&= \frac{1}{2}(N_c - N_f - 2)\left(\frac{16\Lambda_{N_c, N_f}^{3N_c - 6 - N_f}}{\det M}\right)^{1/(N_c - N_f - 2)},
\end{aligned} \tag{4.19}$$

which is the same as the continuation of (3.1) to these values of  $N_c, N_f$ . This guarantees that the expectation values of  $\langle M^{ij} \rangle$  are reproduced correctly when mass terms are added to the magnetic theory.

## 5. $SO(3)$

In this section we discuss the case  $N_c = 3$ , which exhibits some new phenomena. As before, the dual of  $SO(3)$  with  $N_f$  quarks,  $Q^i$ , is an  $SO(N_f + 1)$  theory with  $N_f$  dual quarks,  $q_i$ . The  $N_c = 3$  theory is invariant under the discrete  $Z_{4N_f}$  symmetry (2.4). As in the previous section, the  $Z_{2N_f}$  subgroup acts in the magnetic theory as  $q \rightarrow e^{-2\pi i/2N_f} Cq$ ; the full  $Z_{4N_f}$  should be generated by the “square root” of this operation. The correct

“square root” of the charge conjugation  $\mathcal{C}$  is, as we will show, the  $SL(2, Z)$  electric-magnetic duality modular transformation  $A = TST^2S$ . Therefore, the  $Z_{4N_f}$  symmetry is realized non-locally in the dual theories. In other words, the discrete  $Z_{4N_f}$  symmetry is a “quantum symmetry” in the dual description.

For  $N_c = 3$  a new term has to be added to the dual theory, proportional to

$$\det(q_i \cdot q_j). \quad (5.1)$$

This term can be motivated by several arguments. One of them is by considering the dual of the dual. As we discussed in sect. 4.1, the dual of  $SO(N_f + 1)$  with  $N_f$  flavors is  $SO(3)$  with  $N_f$  flavors with an extra interaction term proportional to  $\det M$ . The term (5.1) is then needed to ensure that the dual of the dual (4.8) is the original theory. For  $N_f \geq 3$  this determines its coefficient:

$$W = \frac{1}{2\mu} M^{ij} q_i \cdot q_j + \frac{1}{64 \tilde{\Lambda}_{N_f+1, N_f}^{2(N_f-1)-1}} \det(q_i \cdot q_j), \quad (5.2)$$

where

$$\epsilon 2^7 \tilde{\Lambda}_{N_f+1, N_f}^{2(N_f-1)-1} \Lambda_{3, N_f}^{3-N_f} = (-1)^{3-N_f} \mu^{N_f}. \quad (5.3)$$

$\epsilon = \pm 1$  arises from taking the square root of the instanton factor  $\Lambda_{3, N_f}^{3-N_f}$  of the electric  $SO(3)$  theory and the phase  $(-1)^{3-N_f}$  keeps the relation (5.3) preserved along the flat directions and with mass perturbations. The superpotential (5.1) is not renormalizable; this will be discussed below.

The term (5.1) is invariant under all the continuous symmetries of the electric theory. However, both magnetic  $SO(N_f + 1)$  instantons (except for  $N_f = 1, 2$ ) and (5.1) break some of the discrete symmetries. Only charge conjugation  $\mathcal{C}$  and the  $Z_{2N_f}$  subgroup of the  $Z_{4N_f}$  symmetry remain unbroken. The underlying  $Z_{4N_f}$  symmetry seems to be explicitly broken. The naive symmetry transformation flips the sign of (5.1) and shifts the theta angle of the magnetic theory (for  $N_f \neq 1, 2$ ) by  $\pi$ . This is consistent with the coefficient in (5.2) and the relation (5.3) as written in terms of the instanton factor  $\tilde{\Lambda}_{N_f+1, N_f}^{2(N_f-1)-1}$  of the magnetic theory. We would like to interpret this as follows. The original electric  $SO(3)$  theory has, in fact, two dual descriptions corresponding to the two signs of this term and a shift of theta by  $\pi$  (for  $N_f \neq 1, 2$ ). One of them is “magnetic,” which was discussed as the “electric” theory in sect. 4.1. The other dual theory is “dyonic.” It will be discussed further in section 6. These two theories are related by another duality transformation,

which extends the group of  $N = 1$  duality transformations to  $SL(2, Z)$ . More precisely, we have only  $S_3 \cong SL(2, Z)/\Gamma(2)$ , which permutes these three theories. The full  $Z_{4N_f}$  symmetry includes the modular transformation which exchanges the magnetic and dyonic theories – it appears as a quantum symmetry in the dual description.

We will now discuss these theories in more detail starting with small values of  $N_f$ .

### 5.1. $N_f = 1$ ; Abelian Coulomb phase and quantum symmetries

This is the  $N = 2$  theory discussed in [5]. Since no superpotential is compatible with the anomaly free  $U(1)_R$  symmetry, the theory has a quantum moduli space of vacua labeled by the expectation value of the massless meson field<sup>5</sup>  $M = Q^2$ . The  $SO(3)$  gauge symmetry is broken to  $SO(2) \cong U(1)$  on this moduli space so the theory has a Coulomb phase with a massless photon, similar to the generic case of  $N_f = N_c - 2$ . The effective gauge coupling of the photon is given by the curve

$$y^2 = x^2(x - M) + 4\Lambda_{3,1}^4 x. \quad (5.4)$$

As discussed in [5], there is a massless magnetic monopole  $q_{(+)}^\pm$  at  $M = 4\Lambda_{3,1}^2$  and a massless dyon  $q_{(-)}^\pm$  at  $M = -4\Lambda_{3,1}^2$ . Their effective superpotentials are

$$W_\pm = f_\pm(M/\Lambda_{3,1}^2) q_{(\pm)}^+ q_{(\pm)}^- \quad (5.5)$$

where in the notation of [5],  $f_+ = a_D(M/\Lambda_{3,1}^2)$  and  $f_- = ia_D(M/\Lambda_{3,1}^2) + ia(M/\Lambda_{3,1}^2)$ , satisfying

$$f_+(-M/\Lambda_{3,1}^2) = f_-(M/\Lambda_{3,1}^2). \quad (5.6)$$

$f_\pm$  is holomorphic around  $M = \pm 4\Lambda_{3,1}^2$  and has a cut along  $[\mp 4\Lambda_{3,1}^2, \infty)$ . Expanding around  $M \approx \pm 4\Lambda_{3,1}^2$ , the superpotentials are

$$W_\pm \approx \frac{1}{2\mu} (M \mp 4\Lambda_{3,1}^2) q_{(\pm)}^+ q_{(\pm)}^- \quad (5.7)$$

where the first term is our standard  $Mq^+q^-$  term and the second term is (5.1).

This theory provides an example of a quantum symmetry. The theory has the global symmetry group  $((SU(2)_R \times Z_8^R)/Z_2) \times \mathcal{C}$ , where the  $Z_8^R$  is an  $R$  symmetry whose generator,  $R$ , acts on all of the  $N = 2$  super charges as a  $e^{2\pi i/8}$  phase and  $\mathcal{C}$  is charge conjugation.

---

<sup>5</sup> Our convention for the normalization of  $\Lambda_{3,1}^2$  differs by a factor of 2 from that of [5] and the order parameter  $u$  of [5] satisfies  $u = \frac{1}{2}M$ .

(The  $Z_{4N_f} = Z_4$  discussed in the introduction of this section is embedded in  $SU(2)_R \times Z_8^R$ .) The  $Z_8^R$  generator acts on the scalar component of  $M$  as  $R : M \rightarrow -M$  and is therefore broken for  $M \neq 0$ . Since  $R^2$  acts as charge conjugation on the squarks, the  $Z_8^R$  symmetry is spontaneously broken to a  $Z_4^R$  generated by  $R^2\mathcal{C}$  for  $M \neq 0$  (alternatively, we could combine the generator of this symmetry with the broken Weyl transformation in the gauge group). At  $M = 0$  the full  $Z_8^R$  symmetry is restored. What is not obvious is that its generator includes an  $SL(2, Z)$  modular transformation,  $w = RA$  with  $A = (TS)^{-1}S(TS)$  [5]. Since  $A^2 = \mathcal{C}$ ,  $w^2 = R^2\mathcal{C}$  generates the  $Z_4$  found away from  $M = 0$ . The necessity of the modular transformation  $A$  in  $w$  can be seen, for example, by considering the central term in the  $N = 2$  algebra,  $Z = a n_e + a_D n_m$  [5]. Since the generator  $R$  in  $w$  multiplies the  $N = 2$  charge by  $e^{2\pi i/8}$ ,  $Z$  must transform under  $w$  as  $w : Z \rightarrow iZ$ . It is easily seen from the integral expressions for  $a(M)$  and  $a_D(M)$  [5] that  $a n'_e + a_D n'_m = i(a n_e + a_D n_m)$  at  $M = 0$  if  $n'_e$  and  $n'_m$  are related to  $n_e$  and  $n_m$  by the modular transformation  $A = (TS)^{-1}S(TS)$ . Note that  $A = \mathcal{C}T(S^{-1}T^2S)$ ; thus,  $A$  is congruent to  $\mathcal{C}T$  modulo multiplication by the monodromy  $S^{-1}T^2S$  associated with looping around one of the singularities.

For  $M \neq 0$  the broken generator  $w$  maps, for example, the massless monopole at  $M = 4\Lambda_{3,1}^2$  to the massless dyon at  $M = -4\Lambda_{3,1}^2$ . At the origin, where the  $Z_8^R$  symmetry is restored, these states are degenerate and are mapped to one another by the symmetry. Since the fields which create these two particles are not relatively local, it is impossible for  $w$  to be given a local realization. Indeed, it is a modular transformation. Furthermore, since  $A$  cannot be diagonalized by an  $SL(2, Z)$  transformation, there is no photon field which is invariant under it. Therefore,  $A$  cannot be realized locally even in the low energy effective Lagrangian at  $M = 0$  which includes only the photon multiplet.

We can now interpret the superpotentials (5.5) as reflecting the symmetries along the lines of the general comments in the introduction to this section and the discussion of the term (5.1). The electric  $SO(3)$  theory has two dual theories. One of them, which we can refer to as the “magnetic dual,” describes the physics around  $M = 4\Lambda_{3,1}^2$  with the superpotential  $W_+$  in (5.5). The other dual, which can be called the “dyonic dual” is valid around  $M = -4\Lambda_{3,1}^2$  and is described by  $W_-$  in (5.5). The magnetic dual is related to the underlying electric theory by the transformation  $S$  in  $SL(2, Z)$  (modulo  $\Gamma(2)$ ) while the dyonic dual is related to the electric description by the  $SL(2, Z)$  transformation  $ST$  (again, modulo  $\Gamma(2)$ ).

Below we will see more complicated examples of quantum symmetries and of magnetic and dyonic duals of the same electric theory.

## 5.2. $N_f = 2$ ; non-Abelian Coulomb and quantum symmetries

As discussed in [7], this theory has three branches described by the superpotentials

$$W = e \frac{\det M}{8\Lambda_{3,2}} + \frac{1}{2} \text{Tr } mM, \quad (5.8)$$

where  $e = 0, \pm 1$  label the three branches. The branch with  $e = 0$  is appropriate for the Higgs or Coulomb phases of the theory. These phases are obtained for  $\det m = 0$ . For  $m = 0$  the generic point in the moduli space is in the Higgs phase. When only  $m_{22}$  is non-zero the low energy theory is that discussed in [5]. It has a massless monopole point at  $M^{11} = 4m_{22}\Lambda_{3,2}$  and a massless dyon point at  $M^{11} = -4m_{22}\Lambda_{3,2}$  (there is an arbitrary choice here in which one is magnetic and which is dyonic). When  $\det m \neq 0$ , the monopole (or the dyon) condenses and leads to confinement (or oblique confinement). Corresponding to these phenomena there are two branches of the theory: a confining branch with  $e = -1$  in (5.8) and an oblique confinement branch with  $e = 1$  in (5.8).

The theory has two dual descriptions in terms of an  $SO(3)$  theory with  $N_f = 2$ :

$$W = \frac{2}{3\mu} \text{Tr } Mq \cdot q + \epsilon \left( \frac{8\tilde{\Lambda}_{3,2}}{3\mu^2} \det M + \frac{1}{24\tilde{\Lambda}_{3,2}} \det q \cdot q \right), \quad (5.9)$$

where  $\epsilon = \pm 1$  labels the two duals and the scales of the theories are related by

$$64\Lambda_{3,2}\tilde{\Lambda}_{3,2} = \mu^2. \quad (5.10)$$

A  $\det M$  term, as in (5.9), was present in the previously discussed  $N_f = N_c - 1$  cases and the  $\det q \cdot q$  term is as in (5.1). In (5.10) we took the square root of a relation involving the instanton factors,  $\Lambda_{3,2}^2$  and  $\tilde{\Lambda}_{3,2}^2$ , of the two groups. The sign ambiguity in doing so is represented in (5.9) by  $\epsilon$ . The coefficients in (5.9) and (5.10) are fixed to guarantee the duality. We will later also determine them by flowing down from other theories.

The  $\det q \cdot q$  term in (5.9) is not renormalizable. This term can be replaced with  $\frac{2}{3\mu} \text{Tr } Lq \cdot q - \frac{32\tilde{\Lambda}_{3,2}^2\epsilon}{3\mu^2} \det L$ , which yields the  $\det q \cdot q$  term in (5.9) upon integrating out  $L$ . The theory with the field  $L$  included has the superpotential

$$W = \frac{2}{3\mu} \text{Tr } (M + L)q \cdot q + \frac{8\epsilon\tilde{\Lambda}_{3,2}}{3\mu^2} (\det M - 4 \det L), \quad (5.11)$$

which is renormalizable.



The electric theory has a  $Z_8$  symmetry, generated by  $Q \rightarrow e^{2\pi i/8}Q$ , and charge conjugation  $\mathcal{C}$ . In the magnetic theory, the  $Z_8$  symmetry of the electric theory takes  $M \rightarrow e^{2\pi i/4}M$  and  $q \rightarrow e^{-2\pi i/8}Aq$ , where  $A$  is a non-local transformation such that  $A^2 = \mathcal{C}$ . We do not have an explicit expression for  $A$  but the consistency of our answers suggests that it exists and hence the  $Z_8$  is a quantum symmetry.

Just as in the electric theory (5.8), the magnetic theory also has three branches with the superpotentials

$$W = \frac{2}{3\mu} \text{Tr } MN + \epsilon \left( \frac{8\tilde{\Lambda}_{3,2}}{3\mu^2} \det M + \frac{1}{24\tilde{\Lambda}_{3,2}} \det N \right) + \tilde{e} \frac{\det N}{8\tilde{\Lambda}_{3,2}}, \quad (5.12)$$

where  $N_{ij} \equiv q_i \cdot q_j$  and, as above,  $\tilde{e} = 0, \pm 1$  labels the three branches.

The superpotential (5.12) is quadratic in both  $M^{ij}$  and  $N_{ij}$ . Therefore,  $M^{ij}$  or  $N_{ij}$  can be integrated out. Integrating out  $N_{ij}$  yields

$$W_{\tilde{e}} = \frac{8\tilde{\Lambda}_{3,2}}{\mu^2} \left( \frac{\tilde{e} - \epsilon}{1 + 3\tilde{e}\epsilon} \right) \det M + \frac{1}{2} \text{Tr } mM = \frac{1}{8\Lambda_{3,2}} \left( \frac{\tilde{e} - \epsilon}{1 + 3\tilde{e}\epsilon} \right) \det M + \frac{1}{2} \text{Tr } mM, \quad (5.13)$$

where we added the  $Q^i$  mass terms  $W_{tree} = \frac{1}{2} \text{Tr } mM$ . This is the same as (5.8) with

$$e = \frac{\tilde{e} - \epsilon}{1 + 3\tilde{e}\epsilon}. \quad (5.14)$$

$\tilde{e} = 0$  describes the weakly coupled Higgs branch of the dual theories. It leads to  $e = -\epsilon$ , which corresponds to the two strongly coupled branches of the electric theory. The Higgs branch of the  $\epsilon = 1$  theory describes the confining branch of the electric theory ( $e = -1$ ) while the Higgs branch of the  $\epsilon = -1$  theory describes the oblique confinement branch of the electric theory ( $e = 1$ ). Therefore, we can refer to the  $\epsilon = 1$  theory as magnetic and to the  $\epsilon = -1$  theory as dyonic. The two other branches of the dual theories are strongly coupled. The branches with  $\tilde{e} = \epsilon$  (oblique confinement of the magnetic theory and confinement of the dyonic theory) lead to  $e = 0$  and therefore to the Higgs branch of the electric theory. Similarly, the branches with  $\tilde{e} = -\epsilon$  (confinement of the magnetic theory and oblique confinement of the dyonic theory) give another description of the strongly coupled branches of the electric theory.

This discussion leads to a new interpretation of the first term in (5.8). In the electric theory this term appears as a consequence of complicated strong coupling dynamics in the confining and the oblique confinement branches of the theory. In the dual descriptions it is already present at tree level.

An equivalent analysis can be performed with the renormalizable theory (5.11). For example, along the flat directions  $q$  gets an effective mass  $\frac{4}{3\mu}(M+L)$  and can be integrated out. The low energy theory is pure-gauge magnetic  $SO(3)$  Yang-Mills theory. The low energy effective superpotential is

$$W_{eff} = \frac{8\epsilon\tilde{\Lambda}_{3,2}}{3\mu^2}(\det M - 4\det L) + \frac{32\eta\tilde{\Lambda}_{3,2}}{9\mu^2}\det(M+L), \quad (5.15)$$

where  $\eta = \pm 1$ . The first terms in (5.15) are the tree-level terms of (5.11) and the last term is generated by gaugino condensation in the magnetic  $SO(3)$  Yang-Mills theory. Integrating out  $L$ , we find

$$W = \frac{8\tilde{\Lambda}_{3,2}}{\mu^2} \left( \frac{\eta\epsilon + 1}{3\epsilon - \eta} \right) \det M. \quad (5.16)$$

Therefore, the flat direction is obtained for  $\eta = -\epsilon$ .

To conclude, we have three equivalent theories: electric, magnetic and dyonic. Every one of them has three branches: Higgs, confinement and oblique confinement. The map between the branches of the different theories is an  $S_3$  permutation described by (5.14).

We now consider the Coulomb phase of the electric theory, obtained by adding  $W_{tree} = \frac{1}{2}mM^{22}$  to (5.12) and integrating out the massive fields by their equations of motion. This gives

$$\begin{aligned} \frac{2}{3\mu}q_2 \cdot q_2 + \frac{8\epsilon\tilde{\Lambda}_{3,2}}{3\mu^2}M^{11} + \frac{1}{2}m &= 0 & M^{22} &= -\frac{\epsilon}{16\tilde{\Lambda}_{3,2}}\mu^{-1}q_1 \cdot q_1 \\ q_1 \cdot q_2 &= 0 & M^{12} &= 0. \end{aligned} \quad (5.17)$$

The expectation value of  $q_2$  breaks the gauge group to  $SO(2)$  for  $M^{11} + 3\epsilon m\mu^2/16\tilde{\Lambda}_{3,2} \neq 0$ . The remaining charged fields,  $q_1^+$  and  $q_1^-$ , couple through the low energy superpotential

$$\frac{1}{2\mu}(M^{11} - m\frac{\epsilon\mu^2}{16\tilde{\Lambda}_{3,2}})q_1^+q_1^- = \frac{1}{2\mu}(M^{11} - 4\epsilon m\Lambda_{3,2})q_1^+q_1^-. \quad (5.18)$$

This superpotential is corrected by contributions from instantons in the broken magnetic  $SO(3)$  theory. For large  $m$  their contribution is small and can be ignored. We see that the theory has massless fields  $q_1^\pm$  at  $M^{11} = 4\epsilon m\Lambda_{3,2} = 4\epsilon\Lambda_{3,1}^2$ . The massless fields for  $\epsilon = 1$  ( $\epsilon = -1$ ) can be interpreted as the monopoles (dyons) of the  $N_f = 1$  theory [5]. We see them as weakly coupled states in the  $\epsilon = 1$  ( $\epsilon = -1$ ) theory. This is in accord with the interpretation of the  $\epsilon = 1$  ( $\epsilon = -1$ ) theory as magnetic (dyonic).

The other monopole point of the  $N_f = 1$  theory arises from strong coupling dynamics in the dual theories. To see that, note that the analysis above is not valid for  $\epsilon M^{11} +$

$12m\Lambda_{3,2} \approx 0$ , where the mass of  $q_1$  is above the Higgs expectation value of  $q_2$ . In that case,  $q_1$  should be integrated out first. For  $u \equiv q_2^2 \neq 0$ , the effective mass of  $q_1$  is  $\frac{4M^{11}}{3\mu} + \frac{\epsilon}{12\Lambda_{3,2}}u$  and the scale of the low energy magnetic theory is thus  $\tilde{\Lambda}_{3,1}^4 = (\frac{4}{3\mu}\tilde{\Lambda}_{3,2}M^{11} + \frac{\epsilon}{12}u)^2$ . There are massless monopoles at  $u = \pm 4\tilde{\Lambda}_{3,1}^2$ , i.e. at  $u = 16\tilde{\Lambda}_{3,2}\mu^{-1}M^{11}/(\pm 3 - \epsilon)$ . The  $M^{22}$  equation of motion in (5.17) gives  $\mu^{-1}u = -(\epsilon/16\Lambda_{3,2})M^{11} - (3/4)m$  and therefore  $M^{11} = 4m\Lambda_{3,2}\frac{\mp 3 + \epsilon}{1 \pm \epsilon}$ . For non-zero  $m$  every value of  $\epsilon$  leads to only one solution, at  $M^{11} = -4\epsilon m\Lambda_{3,2} = -4\epsilon\Lambda_{3,1}^2$ . We have thus found the other monopole of the  $N_f = 1$  theory as a result of strong coupling dynamics in the dual theories.

An analysis similar to the one above for leads to a strongly coupled state in the dual theories along the flat directions with  $\det M = 0$  in the  $m = 0$  case. This state can be interpreted as the massless quark of the electric theory at that point.

Consider taking the dual of the dual theories (5.9). The result is an  $SO(3)$  theory with scale  $\tilde{\tilde{\Lambda}}_{3,2} = \Lambda_{3,2}$ ,  $N_f = 2$  quarks  $d^i$ , gauge singlet fields  $M$  and  $N$ , and a superpotential

$$W = \frac{2}{3\mu} \text{Tr } MN + \epsilon \left( \frac{1}{24\Lambda_{3,2}} \det M + \frac{1}{24\tilde{\tilde{\Lambda}}_{3,2}} \det N \right) - \frac{2}{3\mu} \text{Tr } Ndd + \eta \left( \frac{1}{24\tilde{\tilde{\Lambda}}_{3,2}} \det N + \frac{1}{24\Lambda_{3,2}} \det d \cdot d \right), \quad (5.19)$$

where,  $\epsilon = \pm 1$  and  $\eta = \pm 1$  label the different duals. Using (5.10), the first line in (5.19) is the superpotential associated with the duals (5.9) and the second line are from the duals of that. When  $\epsilon = -\eta$ ,  $N$  is a Lagrange multiplier implementing the constraint  $M = d \cdot d$  and the superpotential is  $W = 0$ . These duals are identified as the original electric theory with  $d^i = Q^i$ . On the other hand, the duals (5.19) with  $\epsilon = \eta$ , upon integrating out  $N$ , have a superpotential

$$W = \frac{\epsilon}{12\Lambda_{3,2}} M^{ij} (d \cdot d)_{ij} - \frac{\epsilon}{24\Lambda_{3,2}} (\det M + \det d \cdot d), \quad (5.20)$$

where we define  $(d \cdot d)_{ij} \equiv \epsilon_{ik}\epsilon_{jl}d^k \cdot d^l$ . These appear to be new dual theories. However, this is not the case. In particular, defining  $q_i \equiv \epsilon_{ij}\sqrt{\epsilon}(\Lambda_{3,2}/\tilde{\tilde{\Lambda}}_{3,2})^{1/4}d^i$  and using (5.10), the superpotential (5.20) is equivalent to the magnetic dual superpotential (5.9). In addition, upon scaling from  $d^i$  to  $q_i$ , the anomaly changes the scale of the theory from  $\Lambda_{3,2}$  to  $\tilde{\tilde{\Lambda}}_{3,2}$ . The duals in (5.20) are, therefore, equivalent to the magnetic duals (5.9). To summarize,  $SO(3)$  with  $N_f = 2$  has three descriptions: the original electric one and the two magnetic duals of (5.9). Taking duals of the duals permutes these three descriptions.

### 5.3. $N_f = 3$

The theory has a bare coupling constant  $\tau_0 = \frac{\theta_0}{2\pi} + \frac{4\pi}{g_0^2}i$  which is not renormalized at one loop. With  $W_{tree} = 0$  the two loop beta function makes the theory not asymptotically free and therefore it is free in the infra-red.

There are dual magnetic and dyonic theories with gauge group  $SO(4) \cong SU(2)_L \times SU(2)_R$  with  $N_f = 3$  flavors  $q_i$  and a superpotential

$$W = \frac{1}{2\mu} \text{Tr } Mqq + \frac{1}{64\tilde{\Lambda}_{s,3}^3} \det qq, \quad (5.21)$$

where the second term is as in (5.2) and the scales  $\tilde{\Lambda}_{s,3}$  of the magnetic  $SU(2)_s$  are equal and are given by

$$\epsilon 2^7 e^{i\pi\tau_0} \tilde{\Lambda}_{s,3}^3 = \mu^3. \quad (5.22)$$

$\epsilon = \pm 1$  reflects the fact that the term  $e^{i\pi\tau_0}$  in (5.22) is the square-root of the  $SO(3)$  instanton factor. The theory is magnetic or dyonic depending on the sign of  $\epsilon$ . In (5.9) and (5.10)  $\epsilon$  appears only in the superpotential and not in the instanton factor of the magnetic group. Here, on the other hand,  $\epsilon$  arises in relating the instanton factor for the magnetic  $SU(2)_s$  to the square root of the instanton factor for the electric  $SO(3)$ .

The analysis of the flat directions with  $W_{tree}$  is similar to that of sect. 4.2. The  $\det qq$  term in (5.21), which was not present in the larger  $N_c = N_f$  theories considered in sect. 4.2, does not significantly modify the analysis.

Consider the theory perturbed by the mass term  $W_{tree} = \frac{1}{2}mM^{33}$ . The electric theory flows to  $N_f = 2$  with a scale  $\Lambda_{3,2}$  given by

$$\Lambda_{3,2}^2 = m^2 e^{2\pi i\tau_0}. \quad (5.23)$$

The threshold factor was determined using our threshold conventions and the results of [6]. Adding the mass term  $W_{tree} = \frac{1}{2}mM^{33}$  to (5.21), the equations of motion give  $\langle q_3^2 \rangle = -\mu m$ , which breaks the magnetic  $SU(2)_L \times SU(2)_R$  gauge group to a diagonally embedded  $SO(3)$ . The relation (5.22) and the matching relations (2.5) and (2.6) imply that the scale  $\tilde{\Lambda}_{3,2}$  of the low-energy magnetic theory is related to the scale  $\Lambda_{3,2}$  of the low-energy electric theory as in (5.10). Integrating out the massive fields at tree level we find

$$W_{tree} = \frac{1}{2\mu} \text{Tr } \hat{M} \hat{q} \hat{q} + \frac{\epsilon}{32\tilde{\Lambda}_{3,2}} \det \hat{q} \hat{q}. \quad (5.24)$$

As in (4.16), we should also include the contribution of instantons in the broken part of the magnetic gauge group. The presence of the non-renormalizable  $\det \hat{q}\hat{q}$  term affects the instantons contributions. Following the discussion which led to (5.11), we replace this term with  $\text{Tr } L\hat{q}\hat{q} - 32\epsilon\tilde{\Lambda}_{3,2} \det L$ , which is the same upon integrating out  $L$ . We can now repeat the analysis leading to (4.16). The effective mass of  $\hat{q}_i$  is  $\frac{1}{\mu}M + 2L$  and therefore the low energy superpotential is

$$W = \frac{1}{2\mu} \text{Tr } (\hat{M} + 2\mu L)\hat{q}\hat{q} - 32\epsilon\tilde{\Lambda}_{3,2} \det L + \frac{2\epsilon\tilde{\Lambda}_{3,2}}{\mu^2} \det(\hat{M} + 2\mu L). \quad (5.25)$$

Integrating out  $L$ , (5.25) yields

$$W = \frac{2}{3\mu} \text{Tr } \hat{M}\hat{q}\hat{q} + \epsilon \left( \frac{8\tilde{\Lambda}_{3,2}}{3\mu^2} \det \hat{M} + \frac{1}{24\tilde{\Lambda}_{3,2}} \det \hat{q}\hat{q} \right), \quad (5.26)$$

the superpotential in (5.9).

#### 5.4. $N_f = 3$ with $W_{tree} = \beta \det Q$ ; $N = 4$ duality as $N = 1$ duality

We now consider perturbing the electric theory by adding the cubic superpotential  $W_{tree} = \beta \det Q$ . For  $\beta = \sqrt{2}$  the theory becomes the  $N = 4$   $SO(3)$  Yang-Mills theory. (We normalize the fields such that the whole Lagrangian has a prefactor of  $\frac{1}{g_0^2}$  and therefore in  $N = 4$  the physical gauge coupling equals the physical Yukawa coupling.) Because of the anomaly, as we rescale  $\beta$  to this value,  $\tau_0$  changes to

$$e^{2\pi i \tau_E} = e^{2\pi i \tau_0} \left( \frac{\beta}{\sqrt{2}} \right)^4 = \frac{1}{4} e^{2\pi i \tau_0} \beta^4. \quad (5.27)$$

After the rescaling the kinetic term of the three  $Q$ 's does not have the proper normalization. However, it is straightforward to see that this feature is achieved in the infra-red. In other words, the theory is attracted to the  $N = 4$   $SO(3)$  Yang-Mills theory in the infra-red with  $\tau_E = \frac{\theta}{2\pi} + \frac{4\pi}{g^2}i$  given by (5.27). An interesting consequence of (5.27) is that the theory only makes sense for  $|\beta| \leq \sqrt{2}|e^{-\frac{1}{2}i\pi\tau_0}|$ .

Using (4.7), we map the electric operator  $\det Q$  to  $(\tilde{W}_\alpha)_L^2 - (\tilde{W}_\alpha)_R^2$  in the magnetic theory. Therefore, adding  $W_{tree} = \beta \det Q$  to the electric theory modifies the magnetic  $SU(2)_L \times SU(2)_R$  theory to have  $\tilde{\Lambda}_{L,3}^3 \neq \tilde{\Lambda}_{R,3}^3$ . The symmetries then determine that the superpotential (5.21) is modified to

$$W = \frac{1}{2\mu} \text{Tr } Mq \cdot q + \frac{2\epsilon e^{i\pi\tau_0}}{\mu^3} f(\tau_E, \epsilon) \det q \cdot q, \quad (5.28)$$

and the scale relation (5.22) is similarly modified to

$$\epsilon 2^7 e^{i\pi\tau_0} \tilde{\Lambda}_{s,3}^3 g_s(\tau_E, \epsilon) = \mu^3 \quad (5.29)$$

where  $f$  and  $g_s$  are functions which we do not determine except to note that for  $\beta = 0$ ,  $\tau_E = i\infty$  and  $f(i\infty, \epsilon) = g_s(i\infty, \epsilon) = 1$ .

In the infra-red, the magnetic  $SU(2)_L \times SU(2)_R$  theory with  $\tilde{\Lambda}_L \neq \tilde{\Lambda}_R$  also flows to an  $N = 4$  theory. This can be seen by considering the  $\tilde{\Lambda}_L \gg \tilde{\Lambda}_R$  limit, corresponding to some value of  $\tau_E$  which we denote by  $\tau_*$ . For  $\tilde{\Lambda}_R = 0$ , the magnetic theory is  $SU(2)_L$  with six doublets coupled through (5.28), which breaks the global  $SU(6)$  to  $SU(3) \times SU(2)_R$  under which the doublets  $q_i$  are in the  $(\bar{3}, 2)$ . The strong  $SU(2)_L$  dynamics confines them to the fields  $N_{ij} = q_i \cdot q_j$ , in the  $(\bar{6}, 1)$ , and  $\phi^i$  in the  $(3, 3)$  of  $SU(3) \times SU(2)_R$ . As in [2] these fields couple through the superpotential  $-\frac{1}{2}\tilde{\Lambda}_L^{-1} N_{ij} \phi^i \cdot \phi^j + \frac{1}{8}\tilde{\Lambda}_L^{-3} \det N + 2 \det \phi$ , where we rescaled  $\phi^i$  to dimension one. Adding this to (5.28) and adding a mass term  $\frac{1}{2}\text{Tr } mM$ , we find the superpotential

$$\frac{1}{2\mu} M^{ij} N_{ij} + \frac{2\epsilon e^{i\pi\tau_0}}{\mu^3} [f(\tau_*, \epsilon) + 2^3 g_L(\tau_*, \epsilon)] \det N - \frac{1}{2}\tilde{\Lambda}_L^{-1} N_{ij} \phi^i \cdot \phi^j + 2 \det \phi + \frac{1}{2}\text{Tr } mM. \quad (5.30)$$

Now we can weakly gauge  $SU(2)_R$ . At energies higher than  $\tilde{\Lambda}_L$ , its coupling constant runs with the scale  $\tilde{\Lambda}_R$ . Below  $\tilde{\Lambda}_L$ ,  $SU(2)_R$  couples to the three triplets  $\phi^i$  and its Wilsonian coupling constant  $\tau_R$  does not run. It thus satisfies

$$e^{2\pi i\tau_R} \sim \frac{\tilde{\Lambda}_R^3}{\tilde{\Lambda}_L^3} \quad (5.31)$$

and hence for  $\tilde{\Lambda}_R \ll \tilde{\Lambda}_L$ ,  $\tau_R \approx i\infty$ .

The fields  $M$  and  $N$  in (5.30) are massive and can be integrated out. The  $M$  equation of motion sets  $N = -\mu m$  and the superpotential (5.30) becomes

$$2 \det \phi + \frac{1}{2} \frac{\mu}{\tilde{\Lambda}_L} m_{ij} \phi^i \cdot \phi^j - 2\epsilon e^{i\pi\tau_0} [f(\tau_*, \epsilon) + 2^3 g_L(\tau_*, \epsilon)] \det m. \quad (5.32)$$

For  $m = 0$  this theory is attracted in the infra-red to an  $SU(2)$   $N = 4$  theory with weak coupling  $\tau_R$ .

Away from the limit  $\tilde{\Lambda}_R \ll \tilde{\Lambda}_L$ , the magnetic theory also flows to an  $N = 4$  theory with some coupling  $\tau_R$  which depends only on  $\tau_E$ .

It is clear that this  $N = 4$  theory with  $\tau_R$  is not the same as the original  $N = 4$  theory with  $\tau_E$  given by (5.27). The original one, with coupling  $\tau_E$ , is weakly coupled for  $\beta \ll 1$ , with  $\tau_E \sim \frac{2}{\pi i} \log \beta$ . The other one, with coupling  $\tau_R$ , is strongly coupled for  $\beta \ll 1$ , where  $\tilde{\Lambda}_L \approx \tilde{\Lambda}_R$ . Conversely, the theory with coupling  $\tau_R$  is weakly coupled when  $\tilde{\Lambda}_L \gg \tilde{\Lambda}_R$ , which happens for  $\beta \sim 1$ , where the original theory is strongly coupled. Although we could not prove that  $\tau_E = -1/\tau_R$ , we suspect that this fact is true and we interpret this theory as being the  $N = 4$  dual of the original theory. This shows that the  $N = 1$  duality of [10] is a generalization of the  $N = 4$  duality of [24].

The meson operator  $M^{ij}$  of the electric theory can be related to the corresponding operator in the  $\tau_R \approx i\infty$  limit of the magnetic theory by differentiating (5.32) with respect to  $m$ :

$$M^{ij} = \frac{\mu}{\tilde{\Lambda}_L} \phi^i \cdot \phi^j - 4\epsilon e^{i\pi\tau_0} [f(\tau_*) + 2^3 g_L(\tau_*)] \det m \left( \frac{1}{m} \right)^{ij}, \quad (5.33)$$

which is not simply proportional to the bilinear  $\phi^i \cdot \phi^j$ . A similar shift was observed in the special case of  $m$  with one vanishing eigenvalue and the two other eigenvalues equal in the flow from  $N = 4$  to  $N = 2$  in [6], thus strengthening our interpretation of the duality map.

## 6. More Dyonic Duals

In sect. 4.1 we found that the theory with  $N_f = N_c - 1$  has a dual magnetic description in terms of an  $SO(3)$  theory with  $N_f$  quarks and the superpotential (4.8). We now consider the dual of this magnetic theory. In sect. 5 we found that  $SO(3)$  theories have both magnetic and dyonic duals. Therefore, there are two duals of the dual of the theory with  $N_f = N_c - 1$ . Both are in terms of an  $SO(N_c)$  gauge theory with  $N_f$  matter fields  $d^i$  and gauge singlet fields  $M^{ij}$  and  $N_{ij}$  with a superpotential

$$W = \frac{1}{2\mu} \text{Tr } N(M - dd) - \frac{1}{64\Lambda_{N_c, N_c-1}^{2(N_c-2)-1}} (\det M - \epsilon \det dd), \quad (6.1)$$

with  $\epsilon = \pm 1$ , and scales

$$\tilde{\Lambda}_{N_c, N_c-1}^{2N_c-5} = \epsilon \Lambda_{N_c, N_c-1}^{2N_c-5}. \quad (6.2)$$

The  $N$  equation of motion of (6.1) gives  $M^{ij} = d^i \cdot d^j$ . The theory (6.1) with  $\epsilon = 1$ , the magnetic dual of the magnetic dual, gives  $W = 0$ ; this is the original electric theory with  $d^i$  identified with  $Q^i$ . On the other hand, the theory (6.1) with  $\epsilon = -1$  has a superpotential

$$W = -\frac{1}{32\Lambda_{N_c, N_c-1}^{2N_c-5}} \det d \cdot d = \frac{1}{32\tilde{\Lambda}_{N_c, N_c-1}^{2N_c-5}} \det d \cdot d \quad (6.3)$$

and, from (6.2), a theta angle differing from that of the original electric theory by a shift by  $\pi$ . We will refer to this theory as the “dyonic dual” of the original theory.

Near the origin in field space the operator (6.3) is irrelevant and does not affect the dynamics. The flat directions of this theory are more subtle. Analyzing the theory classically we might conclude that the moduli space of vacua is given by all values of  $M^{ij} = d^i \cdot d^j$  subject to  $\det M = 0$ , which breaks the gauge group to  $U(1)$ . However, as we move away from the origin we face the following problem. Consider the direction in field space where  $M$  is diagonal and has  $N_f - 1$  non-zero equal eigenvalues  $a$ . For  $a \gg \Lambda^2$  some quarks acquire masses of order  $a^{N_f-1}/\Lambda^{2N_f-3}$  while the massive gauge bosons are much lighter; their mass is of order  $\sqrt{a}$ . In the energy range between these two values the gauge group is not broken but the quarks are not in  $SO(N_c)$  representations. This happens because the interaction (6.3) is not renormalizable. Therefore, it cannot be used for large  $a$ . Equivalently, in the limit of large  $d$  the gauge symmetry is broken at a high scale and the gauge interactions are weakly coupled. However, the superpotential (6.3) leads to strong coupling for the massive fields. Therefore, they cannot be easily integrated out and the classical analysis is misleading.

Near the origin we can analyze the flat directions by first neglecting (6.3). Then, the theory is similar to the electric theory and has several branches. Its oblique confinement branch is described by the superpotential (4.13) in the theory with scale  $\tilde{\Lambda}$ ; this  $W_{oblique}$  differs from (6.3) by a sign. Adding  $W_{oblique}$  to (6.3) gives  $W = 0$ . In this branch of the dyonic theory we thereby recover the flat directions, given by the space of  $M^{ij}$ , exactly as in the electric theory, except that in this theory it has a strongly coupled description.

Consider perturbing the dyonic theory (6.3) by  $W_{tree} = \frac{1}{2}mM^{N_f N_f}$ . Near the origin the dynamics is strongly coupled, as in the electric theory, and we find the multi-monopole point at strong coupling. Away from the origin (for  $m \ll \Lambda$ ) we can integrate out the massive fields. Their equations of motion give  $\hat{d}^i \cdot d^{N_f} = d^{N_f} \cdot d^{N_f} = 0$  for  $\hat{i} = 1 \dots N_c - 2$ , which generically break  $SO(N_c)$  to  $SO(2)$ . The massless fields are  $\hat{M}^{\hat{i}\hat{j}} = \hat{d}^i \cdot \hat{d}^j$ . However, in the region  $\det \hat{M} \approx 16\Lambda_{N_c, N_c-2}^{2(N_c-2)}$ , there are also light charged fields  $d^\pm$  coming from  $d^{N_f}$ . The superpotential in the low energy theory is

$$W = \frac{1}{2}m\left(1 - \frac{\det \hat{M}}{16\Lambda_{N_c, N_c-2}^{2(N_c-2)}}\right)d^+d^-, \quad (6.4)$$

showing that the charged fields  $d^\pm$  are massless at  $\det \hat{M} = 16\Lambda_{N_c, N_c-2}^{2(N_c-2)}$ . The fields  $d^\pm$  can be interpreted as the dyons  $E^\pm$  of the low energy  $N_f = N_c - 2$  theory. These dyons



were found in sect. 3.4 by means of a strong coupling analysis of the electric theory and in sect. 4.1 by a strong coupling analysis in the magnetic theory. Here we find these fields in a weak coupling analysis of the dyonic theory. This gives a new interpretation of the oblique confining superpotential (4.13) – it is present in the tree level Lagrangian of the dyonic theory (6.3).

Taking the magnetic dual of the dyonic dual (6.3) gives an  $SO(3)$  theory with  $N_f$  quarks and the superpotential

$$W = \frac{1}{2\mu} M^{ij} q_i \cdot q_j - \frac{1}{\tilde{\Lambda}_{N_c, N_c-1}^{2N_c-5}} \det M + \frac{1}{32\tilde{\Lambda}_{N_c, N_c-1}^{2N_c-5}} \det M, \quad (6.5)$$

where the first two terms are as in (4.8) and the last term is the tree level term (6.3) of the dyonic theory. This magnetic theory is the same as the one in (4.8); in particular, using (6.2) and (4.9), the scale and superpotential are the same. Taking the dyonic dual of the dyonic dual (6.3) shifts the theta angle by  $\pi$  again and gives a superpotential which cancels (6.3); this gives back the original electric theory. To summarize, the  $SO(N_c)$  theory with  $N_c - 1$  flavors has three descriptions: the original electric one, the magnetic  $SO(3)$  one discussed in sect. 4.1, and the dyonic  $SO(N_c)$  one with theta angle shifted by  $\pi$  and the superpotential (6.3). Taking duals of duals permutes these three descriptions.

### Acknowledgments

We would like to thank T. Banks, R. Leigh, M.R. Plesser, P. Pouliot, S. Shenker, M. Strassler and E. Witten for useful discussions. This work was supported in part by DOE grant #DE-FG05-90ER40559.

## References

- [1] N. Seiberg, hep-ph/9309335, Phys. Lett. 318B (1993) 469
- [2] N. Seiberg, hep-th/9402044, Phys. Rev. D49 (1994) 6857
- [3] V. Kaplunovsky and J. Louis, hep-th/9402005, Nucl. Phys. B422 (1994) 57
- [4] K. Intriligator, R.G. Leigh and N. Seiberg, hep-th/9403198, Phys. Rev. D50 (1994) 1092; K. Intriligator, hep-th/9407106, Phys. Lett. 336B (1994) 409
- [5] N. Seiberg and E. Witten, hep-th/9407087, Nucl. Phys. B426 (1994) 19
- [6] N. Seiberg and E. Witten, hep-th/9408099, Nucl. Phys. B431 (1994) 484
- [7] K. Intriligator and N. Seiberg, hep-th/9408155, Nucl. Phys. B431 (1994) 551
- [8] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, hep-th/9411048, hep-th/9412158; P. Argyres and A. Faraggi, hep-th/9411057
- [9] K. Intriligator, N. Seiberg and S. Shenker, hep-ph/9410203, Phys. Lett. 342B (1995) 152
- [10] N. Seiberg, hep-th/9411149, RU-94-82, IASSNS-HEP-94/98
- [11] O. Aharony, hep-th/9502013, TAUP-2232-95
- [12] D. Kutasov, hep-th/9503086, EFI-95-11
- [13] R. Leigh and M. Strassler, RU-95-2, hep-th/9503121
- [14] M. Douglas and S. Shenker, RU-95-12, to appear
- [15] D. Finnell and P. Pouliot, RU-95-14, SLAC-PUB-95-6768, hep-th/9503115
- [16] N. Seiberg, The Power of Holomorphy – Exact Results in 4D SUSY Field Theories. To appear in the Proc. of PASCOS 94. hep-th/9408013, RU-94-64, IASSNS-HEP-94/57
- [17] I. Affleck, M. Dine and N. Seiberg, Nucl. Phys. B241 (1984) 493; Nucl. Phys. B256 (1985) 557
- [18] V.A. Novikov, M.A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B223 (1983) 445; Nucl. Phys. B260 (1985) 157
- [19] V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, Nucl. Phys. B229 (1983) 381
- [20] D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, Phys. Rep. 162 (1988) 169 and references therein
- [21] T. Banks, E. Rabinovici, Nucl. Phys. B160 (1979) 349; E. Fradkin and S. Shenker, Phys. Rev. D19 (1979) 3682.
- [22] G. 'tHooft, Nucl. Phys. B190 (1981) 455
- [23] J. Cardy and E. Rabinovici, Nucl. Phys. B205 (1982) 1; J. Cardy, Nucl. Phys. B205 (1982) 17
- [24] C. Montonen and D. Olive, Phys. Lett. 72B (1977) 117; P. Goddard, J. Nuyts and D. Olive, Nucl. Phys. B125 (1977) 1
- [25] H. Osborn, Phys. Lett. 83B (1979) 321; A. Sen, hep-th/9402032, Phys. Lett. 329B (1994) 217; C. Vafa and E. Witten, hep-th/9408074, Nucl. Phys. B432 (1994) 3